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Properties of some five dimensional Einstein metrics

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Abstract

The volumes, spectra and geodesics of a recently constructed infinite family of five-dimensional inhomogeneous Einstein metrics on the two S^3 bundles over S^2 are examined. The metrics are in general of cohomogeneity one but they contain the infinite family of homogeneous metrics $T^{p,1}$. The geodesic flow is shown to be completely integrable, in fact both the Hamilton-Jacobi and the Laplace equation separate. As an application of these results, we compute the zeta function of the Laplace operator on $T^{p,1}$ for large p. We discuss the spectrum of the Lichnerowicz operator on symmetric transverse tracefree second rank tensor fields, with application to the stability of Freund-Rubin compactifications and generalised black holes.

Contents

1 Introduction

Compact Riemannian Einstein manifolds may be used as basic building blocks for solutions to higher dimensional gravity and supergravity theories. An important example in recent years has been Freund-Rubin compactification in the context of the AdS/CFT correspondence. Here one has backgrounds such as $AdS_5 \times M^5$ supported by 5-form flux. The Einstein manifold M encodes geometrically the properties of the dual conformal field theory such as the R-symmetry and central charge [1]. Another set of examples are generalised D-dimensional black holes, where the horizon is given by an arbitrary Einstein manifold M rather than the usual sphere S^{D-2}[2] . In fact, the two examples we have just given are related. A generalised cone spacetime over the Einstein manifold M is a Ricci flat Lorentzian metric

$$ds^2 = -dt^2 + dr^2 + r^2 ds_M^2 \,. \tag{1}$$

Generalised black holes may be thought of as black holes in generalised cone spacetimes, see (20) below. Instead of black holes, we could have appended three more flat directions and considered an extremal D3-brane sitting at the tip of the cone. The near horizon geometry of this D3 brane is then $AdS_5 \times M^5$ [3].

Homogeneous Einstein manifolds have been known for some time, well-known examples in the physics literature include the round spheres S^d and the five dimensional $T^{p,q}$ spaces. It is harder to find explicit inhomogeneous Einstein metrics on compact manifolds. When a metric is found, it is then useful to study its properties, both to achieve a better geometric understanding of the manifold and with a view to physical applications. One key question for inhomogeneous manifolds is the separability of partial differential equations such as the Laplace equation and the Hamilton-Jacobi equation for geodesics. Separability is an important first step for being able to perform calculations involving the metrics.

In four dimensions, the first explicit inhomogeneous compact Einstein metric was constructed by Page [4]. Topologically the manifold is the nontrivial S^2 bundle over S^2, which is isomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. The method of [4] was generalised recently to obtain, amongst other results, an infinite series of inhomogeneous Einstein metrics in five dimensions on $S^2 \times S^3$ and on the nontrivial S^3 bundle over S^2 [5]. The infinite series is parameterised by two integers (k_1, k_2). When $k_1 = k_2 \equiv k$, the metrics become the series of homogeneous

metrics $T^{k,1}$. This construction was then further generalised to higher dimensions in [6]. In this paper we shall be concerned with the properties of the five dimensional metrics labelled by (k_1, k_2) [5].

There have been two other recent explicit contructions of inhomogeneous Einstein manifolds. Firstly, Böhm has constructed an infinite family of inhomogeneous metrics on $S^5 \cdots S^9$ and products of spheres [7]. These metrics can be unwieldy because the metric functions are not given explicitly, but as solutions to nonlinear Ordinary Differential Equations. However, some properties are known[8]. Secondly, an explicit infinite family of inhomogeneous Einstein-Sasaki manifolds were recently constructed on $S^2 \times S^3$ in five dimensions [9] and also generalised to higher dimensions [10].

An important question concerning Freund-Rubin compactifications and generalised black hole spacetimes is whether they are stable. In both cases, the question of stability reduces to the question of whether the spectrum of the Lichnerowicz operator acting on rank two symmetric tensor fields satisfies a certain lower bound [11, 2, 8, 12]. For the AdS/CFT correspondence one is more interested in stable spacetimes, as this is when one expects a valid duality. For generalised black holes spacetimes, one may also be interested in unstable spacetimes because the endpoint of the instability is a nontrivial and interesting dynamical question in higher dimensional relativity. Finally, one may also consider the stability of generalised black holes with a negative cosmological constant [12]. In this case, the stability or instability of the spacetime is related to a poorly understood phase transition in a dual thermal field theory induced by the inhomogeneity of the background [12, 13].

1.1 Outline of the paper

The organisation of this paper is as follows.

In section 2 below, we review the Einstein metrics constructed in [5]. We give a broad picture of the discrete moduli space of metrics and we write the metrics in a form where the $SU(2) \times U(1)$ isometry and the cohomogenity one property is manifest. We then go on to characterise the moduli space in terms of volumes and Weyl curvature eigenvalues. The latter allows us to comment on the stability and instability of the resulting spacetimes.

In section 3 we study the Laplacian spectrum on the manifolds. In the homogeneous cases, $T^{p,1}$, we give the spectrum explicitly. The spectrum on $T^{p,q}$ has previously been studied in [1]. In the inhomogeneous case we show that the Laplace equation separates and for slightly inhomogeneous metrics we give the spectrum as a perturbation about the homogeneous cases.

In section 4 we use the results on the Laplacian spectrum to calculate the zeta function on $T^{p,1}$ for large p. The zeta function contains information such as the thermodynamics of a scalar gas on the Einstein background. Ultimately, one would like to calculate the zeta function for an inhomogeneous background, as the corresponding thermodynamics may provide some insight into the phase transition expected on backgrounds with a region of large curvature [13].

In section 5 we study the geodesics of the Einstein metrics. We show that the Hamilton-Jacobi equation separates and describe some qualitative features of particle motion. For the homogeneous metrics we give a full action-angle analysis of the geodesics. This allows us to calculate the periods of geodesics and the semiclassical energy spectrum. The semicalssical spectrum agrees with the full spectrum of the Laplacian up to ordering ambiguities in quantisation. In the inhomogeneous case the action-angle problem reduces to an evaluation of elliptic integrals. These may be calculated in a perturbation about the homogeneous metrics, and again the semiclassical energy spectrum has a good agreement with the full Laplacian spectrum.

Section 6 is the conclusion and contains suggestions for future research.

2   The metrics

The complete Einstein metrics presented in [5] depend upon two integers k_1 and k_2 which are the Chern numbers of a principle T^2 bundle over S^2. The 5-manifold is then an associated S^3 bundle over S^2. Topologically there are two such bundles because $\pi_1(SO(4)) = \mathbb{Z}_2$. If $k_1 + k_2$ is even the bundle is trivial and if $k_1 + k_2$ is odd the bundle is non-trivial.

The metrics may be written as

$$ds_5^2 = h(\theta)^2 d\theta^2 + b(\theta)^2 \left(d\chi^2 + \sin^2 \chi d\eta^2\right) + a_{ij}(\theta)(d\psi^i + \cos \chi d\eta)(d\psi^j + \cos \chi d\eta), \quad (2)$$

where the ranges of the angles are $0 \leq \theta \leq \pi/2$, $0 \leq \chi \leq \pi$, $0 \leq \eta < 2\pi$ and $0 \leq \psi^i < 4\pi/|k_i|$.

The general expressions for the metric quantities h, b, a_{ij} and the integers k_i are given in appendix A. Throughout we will use a normalisation such that the Ricci scalar of the metric is $R = 20$. The explicit formulae for the metric depend upon two constants ν_1 and ν_2 which depend in a complicated implicit fashion (given in appendix A) on the integers k_1 and k_2. We shall be concerned with the case when $\nu_1, \nu_2 > 1$. In this case it appears that for each pair of positive integers k_1, k_2 there is a unique pair $\nu_1, \nu_2 > 1$. Note that there is a symmetry interchanging k_1 and k_2 and with it ν_1 and ν_2.

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We will shortly consider the moduli space of Einstien metrics in some detail. Let us first exhibit some special cases. A schematic summary of the following statements is contained in Figure 1 below.

• If $k_1 = k_2 = k$, hence $\nu_1 = \nu_2 = \nu$, we obtain the homogeneous metrics known in the physics literature as $T^{k,1}$. After the change of variables [5]

$$\beta = 2\theta, \qquad \gamma = \frac{1}{2}(\psi^2 - \psi^1), \qquad t = \frac{1}{2}(\psi^1 + \psi^2), \tag{3}$$

the metrics takes the standard form for $T^{k,1}$

$$
\begin{aligned}
ds^2_{T^{k,1}} &= \frac{1+\nu^2}{4(2+\nu^2)}\left(d\beta^2 + \sin^2\beta d\gamma^2\right) + \frac{1+\nu^2}{4(2\nu^2+1)}\left(d\chi^2 + \sin^2\chi d\eta^2\right) \\
&+ \frac{1+\nu^2}{2(2+\nu^2)^2}\left(dt + \cos\beta d\gamma + k\cos\chi d\eta\right)^2,
\end{aligned}
\tag{4}
$$

where $k = \nu(\nu^2 + 2)/(2\nu^2 + 1)$. The case $(k_1, k_2) = (1,1)$ is the Einstein-Sasaki metric known as $T^{1,1}$.

• The cases $(k_1, k_2) = (0,2)$ or $(2,0)$ have $\nu_1 = 1$ or $\nu_2 = 1$ and ν_2 or ν_1 arbitrary, respectively. These are on the boundary of the cases we consider. The metric is independent of ν_2 or ν_1 and coincides with the round metric on $S^5/\mathbb{Z}_2$.

•. We can let $\nu_2 \to \infty$ with ν_1 finite. One finds that

$$\nu_1 \to \frac{k_1}{2} + \sqrt{\frac{k_1^2}{4} - 1}, \qquad k_2 \to 0. \tag{5}$$

We call this the vertical limit. The metric approaches the round metric on $S^5/\mathbb{Z}_{k_1}$. It has an orbifold singularity along a circle. This may be described as follows. The metric takes the form

$$ds^2_\infty = d\theta^2 + \sin^2\theta\, ds^2_3 + \cos^2\theta d\psi^2, \tag{6}$$

where ψ has range 2π and

$$ds^2_3 = \frac{1}{4}\left[(d\psi_1 + \cos\chi d\eta)^2 + d\chi^2 + \sin^2\chi d\eta^2\right]. \tag{7}$$

The angle ψ_1 is identified modulo $\frac{4\pi}{k_1}$. Therefore, there is an orbifold singularity along the circle at $\theta = 0$. Locally the singularity is $\mathbb{R}^4/\mathbb{Z}_{k_1} \times S^1$.

A word of caution about the vertical limit is necessary. Given that $k_2 \to 0$ and k_2 is an integer, in fact the only solution is $k_2 = 0$. Thus, although the limiting orbifold metrics we have just described exist, there are no Einstein metrics 'near' the limiting metric. A more interesting limit is the limit in which we allow both ν_1 and ν_2 to become large.

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- For large values of ν_1 and ν_2 we find that

$$\nu_1 \approx \frac{1}{k_1}(k_1^2 + k_2^2), \qquad \nu_2 \approx \frac{1}{k_2}(k_1^2 + k_2^2).$$

(8)

We call the limit in which both ν_1 and ν_2 become large, with the ratio $\nu_2/\nu_1 \sim k_1/k_2 \equiv q$ fixed, the rational limit. Near the rational limit, the metric becomes

$$ds^2 \approx d\theta^2 + \frac{\cos^2\theta + q^2\sin^2\theta}{4(1+q^2)} ds_{S^2}^2 + \frac{\nu_2^2 q^2 \sin^2\theta \cos^2\theta(\omega_1 - \omega_2)^2 + (1+q^2)(q^4\sin^4\theta\,\omega_1^2 + \cos^4\theta\,\omega_2^2)}{4(1+q^2)^2(\cos^2\theta + q^2\sin^2\theta)},$$

(9)

where $\omega_i = d\psi^i + \cos\chi d\eta$. The actual limiting metric itself is not necessarily Einstein in this limit. However, we do have an infinite sequence of Einstein metrics as we approach the limit. Therefore, this limit is a richer source of Einstein metrics than the vertical limit.
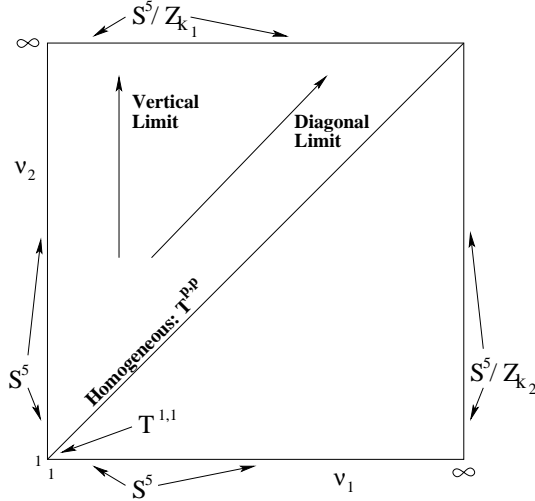

Figure 1: Schematic representation of the limits in the moduli space of Einstein metrics. The figure is a little misleading in that the rational limit should tend to the upper right hand corner, and there are no metrics near the vertical limit.

2.1 The isometry group

An $SU(2)_L \times U(1)_L \times U(1)_R$ isometry of the metric may be made manifest by rewriting the metric in terms of left-invariant $SU(2) \times U(1)$ forms

$$\sigma_1 = \cos\left(\frac{\psi^1 + \psi^2}{2}\right) d\chi + \sin\left(\frac{\psi^1 + \psi^2}{2}\right)\sin\chi d\eta,$$

$$\sigma_2 = -\sin\left(\frac{\psi^1 + \psi^2}{2}\right) d\chi + \cos\left(\frac{\psi^1 + \psi^2}{2}\right)\sin\chi d\eta,$$

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$$\sigma_3 = \frac{1}{2}(d\psi_1 + d\psi_2) + \cos\chi d\eta\,,$$

$$\sigma_4 = \frac{1}{2}(d\psi_1 - d\psi_2)\,. \tag{10}$$

The metric becomes

$$ds_5^2 = h^2 d\theta^2 + b^2\left(\sigma_1^2 + \sigma_2^2\right) + a_{11}(\sigma_3 + \sigma_4)^2 + a_{22}(\sigma_3 - \sigma_4)^2 + 2a_{12}(\sigma_3 - \sigma_4)(\sigma_3 + \sigma_4)\,, \tag{11}$$

which we write as

$$\boxed{ds_5^2 = h^2 d\theta^2 + b^2\left(\sigma_1^2 + \sigma_2^2\right) + c^2\left(f\sigma_3 + \sigma_4\right)^2 + g^2\sigma_4^2\,,} \tag{12}$$

with

$$c^2 = \frac{(a_{22} - a_{11})^2}{a_{11} + a_{22} + 2a_{12}}\,, \qquad g^2 = \frac{4(a_{22}a_{11} - a_{12}^2)}{a_{11} + a_{22} + 2a_{12}}\,, \qquad f = \frac{a_{11} + a_{22} + 2a_{12}}{a_{11} - a_{22}}\,. \tag{13}$$

The second form of the metric written here is useful for calculation using the obvious vielbein

$$e^1 = hd\theta\,, \qquad e^2 = b\sigma_1\,, \qquad e^3 = b\sigma_2\,, \qquad e^4 = c(f\sigma_3 + \sigma_4)\,, \qquad e^5 = g\sigma_4\,. \tag{14}$$

The left-acting isometries follow from writing the metric in terms of left-invariant forms. The remaining $U(1)$ symmetry is the usual right-acting $U(1)$ isometry of the squashed three-sphere. It is present because the metric does not depend on η.

2.2 Moduli space of metrics: Volumes and Weyl eigenvalues

Figure 2 shows the discrete moduli space of Einstein metrics with $1 < \nu_1, \nu_2 < 100$. Each point corresponds to an Einstein metric.

So far we have parameterised the moduli space in terms of ν_1, ν_2 or k_1, k_2. Although these quantities were useful for discussing various limits, they do not have an invariant geometric meaning. In this subsection we will consider how two geometric properties of the metrics vary as we move around the moduli space. The two properties will be the volume of the manifold and the eigenvalues of the Weyl tensor. The reason for choosing these quantities is that they are related to the physics of generalised black holes and Freund-Rubin compactifications constructed using these metrics. We will elaborate on physical implications in the next subsection.

The volume of the manifolds is straightforward to calculate and is given by

$$\mathrm{Vol}(\nu_1, \nu_2) = \frac{(\nu_1^2\nu_2^2 - 1)^{5/2}(\nu_1^2 + \nu_2^2 - 2)^{3/2}\pi^3}{(1 + \nu_1^4\nu_2^2 + \nu_1^2\nu_2^4 - 3\nu_1^2\nu_2^2)(\nu_2^2 + \nu_1^2\nu_2^2 - 2)(\nu_1^2 + \nu_1^2\nu_2^2 - 2)}\,. \tag{15}$$
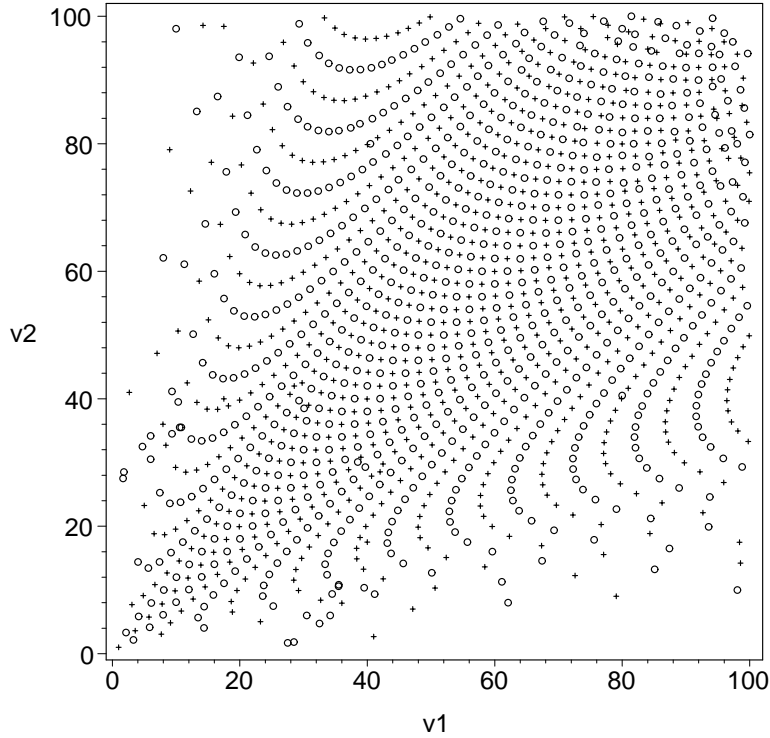
Figure 2: Moduli space of Einstein metrics. Crosses correspond to topology $S^3 \times S^2$. Circles have the topology of the nontrivial S^3 bundle over S^2.

The eigenvalues of the Weyl tensor acting on symmetric tracefree tensors are given by

$$C_a{}^c{}_b{}^d h_{cd} = \kappa h_{ab}. \tag{16}$$

For inhomogeneous manifolds the eigenvalues $\kappa = \kappa(x)$ depend on the position. The maximum value taken by an eigenvalue on the manifold, κ_0, gives a lower bound for the Lichnerowicz spectrum[8]. In five dimensions and with our normalisation for the Einstein metrics, the bound is

$$\Delta_L \geq 4(5 - \kappa_0). \tag{17}$$

The classical stability of generalised black holes and Freund-Rubin compactifications depends on the Lichnerowcz spectrum, so κ_0 is an interesting quantity to consider.

Let us assume that $\nu_2 \geq \nu_1$. Results for the opposite case may be obtained by interchanging ν_1 and ν_2. In the present case, it turns out that the maximum value of the Weyl eigenvalues is achieved at $\theta = 0$ and may be shown to be given by a fairly simple expression

$$\kappa_0 = \frac{(\nu_2^2 - 1)^2 (2 + 2\nu_1^2 + \sqrt{9 - 2\nu_1^2 + 9\nu_1^4})}{(\nu_1^2 - 1)(\nu_1^2 \nu_2^2 - 1)}. \tag{18}$$

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To derive this expression, one should work with an orthonormal set of vielbeins (14) and consider a basis of fourteen symmetric tracefree matrices. The action of the Weyl tensor (16) on this basis will then define a matrix whose eigenvalues are required.

We can gain some intuition about instabilities by considering the eigenmode corresponding to the eigenvalue (18). At $\theta = 0$, and using the tangent space basis given by the vielbeins (14) we have

$$
h^0_{\bar{a}\bar{b}} = \begin{pmatrix} 4 & & & & \\ & -3+x & & & \\ & & -3+x & & \\ & & & -2-2x & \\ & & & & 4 \end{pmatrix}, \tag{19}
$$

where $x = 3\nu_1^2 - \sqrt{9 - 2\nu_1^2 + 9\nu_1^4}$. One may check that $C_a{}^c{}_b{}^d h^0_{cd} = \kappa_0 h^0_{ab}$ at $\theta = 0$. We expect that for the unstable spacetimes the unstable mode will be concentrated near $\theta = 0$ and will be well approximated by h^0 at that point. It is curious that the mode does not depend on ν_2.

Table 1 shows the values of the volume and the maximum Weyl eigenvalue in two limits.

Limit	Volume	Weyl eigenvalue
Homogeneous: $\nu_1 = \nu_2 = \nu$	$\dfrac{(1+\nu^2)^{5/2} 2^{3/2}\pi^3}{(2+\nu^2)^2(2\nu^2+1)}$	$\dfrac{2 + 2\nu^2 + \sqrt{9 - 2\nu^2 + 9\nu^4}}{1+\nu^2}$
'Rational limit': $\begin{aligned} &\nu_1, \nu_2 \to \infty \\ &\nu_2/\nu_1 = q \geq 1 \text{ fixed} \end{aligned}$	$\sqrt{1 + \dfrac{1}{q^2}}\dfrac{\pi^3}{\nu_1}$	$5q^2$

Table 1: Volumes and maximum of Weyl eigenvalues in various limits.

We see that in the rational limits, including the homogeneous limit $\nu \to \infty$, the volume goes to zero. On the other hand, the Weyl eigenvalue remains finite and may be tuned to any value allowed by the integrality constraints: $k_i \in \mathbb{Z}^+$.

2.3 Physical consequences: Entropy and stability

Five dimensional compact Einstein manifolds appear in a simple way in two contexts in higher dimensional gravity. Firstly, they can be used to construct generalised black hole spacetimes in seven dimensions

$$
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 ds_5^2 \tag{20}
$$

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with $f(r) = 1 - (1 + r_+^2/L^2)(r_+/r)^4 + r^2/L^2$. This is a solution to Einstein's equations, with a possible negative cosmological constant $-1/L^2$. The horizon is at $r = r_+$. The volume of the five dimensional manifold becomes the area of the event horizon and is therefore proportional to the entropy of the black holes.

The classical stability of generalised black holes has been studied recently, both with vanishing cosmological constant [2, 8] and with a negative cosmological constant [12]. The latter case has interesting field theory implications using the AdS/CFT correspondence [13].

For generalised black holes there is a simple criterion for classical stability. The criterion depends on the minimum Lichnerowicz eigenvalue of the horizon. For a five dimensional horizon, and with vanishing cosmological constant, the criterion is [2]

$$\Delta_L \geq 4 \qquad \Leftrightarrow \qquad \text{stability} \,. \tag{21}$$

Therefore, from (17) we see that if any of the metrics have $\kappa_0 \leq 4$, they will result in stable spacetimes.

For large generalised black holes with a negative cosmological constant, $r_+/L \gg 1$, the stability criterion is [12]

$$\Delta_L \geq -A^2 \times \frac{r_+^2}{L^2} \qquad \Leftrightarrow \qquad \text{stability} \,, \tag{22}$$

where A^2 is a positive $\mathcal{O}(1)$ coefficient that may be determined numerically [12]. In order to be unstable, the maximum Weyl eigenvalue κ_0 must therefore be very large.

A second type of solution that uses Einstein manifolds are Freund-Rubin compactifications with a 5 form field strength

$$
\begin{aligned}
ds^2 &= ds^2_{\text{AdS}_{D-5}} + ds_5^2, \\
F_5 &= \left[\frac{8(D-2)}{D-6} \right]^{1/2} \text{vol}(M^5) \,.
\end{aligned}
\tag{23}
$$

Remarkably the stability of these spacetimes [11, 8] is according to precisely the same criterion as for generalised black holes (21). This may be due to the relation between generalised black holes and Freund-Rubin compactifications pointed out in the introduction.

The relation between the Lichnerowicz spectrum and the eigenvalues of the Weyl tensor (17) is such that it allows us to prove the stability of the above spacetimes, but not instability. However, it was found in [8] that large positive values of κ_0 tend to suggest unstable spacetimes.

One can check from (18) that $\kappa_0 > 4$ for all values except $\nu_1 = \nu_2 = \nu = 1$, which has precisely $\kappa_0 = 4$. Therefore the only rigorous statement about stability we can make is that

the $\nu = 1$ metric gives marginally stable spacetimes. This metric is non other than $T^{1,1}$ so we have reproduced a known fact [14]. However, it seems very likely that all the remaining metrics will give unstable spacetimes. The instability means that these metrics have limited interest for the AdS/CFT correspondence, but give rise to interesting physics of generalised black holes.

Like the Böhm metrics [8, 7], the moduli space here includes regions where the maximum Weyl eigenvalue is becoming arbitrarily large. For example, q is arbitrary in Table 1. This suggests that the lower bound on the Lichnerowicz spectrum may become arbitrarily negative. This in turn implies that the metrics give not only unstable flat space black holes, but also unstable large black holes in anti-de Sitter space according to the criterion (22).

Unlike the Böhm metrics however, the metrics we consider here are given explicitly, without need of numerical calculations. Therefore it is substantially easier to study quantum fields on these backgrounds. We begin this study below by computing the zeta function of a free scalar field in some cases. Ultimately, one would like to understand better the nature of the field theory instability that seems to appear when the maximum Weyl eigenvalue becomes large [13].

3   Laplacian spectrum

In this section we show how the Laplace eigenvalue equation for the metrics we are considering separates. In the homogeneous cases we find the spectrum explicitly. The equation we need to solve is

$$-\Box\phi = \lambda\phi \,, \tag{24}$$

with

$$\Box = \Box_\theta + \frac{1}{b(\theta)^2 \sin\chi} \frac{\partial}{\partial\chi} \sin\chi \frac{\partial}{\partial\chi} + \frac{1}{b(\theta)^2} \left(e_3\right)^2 + a^{ij}(\theta) \frac{\partial}{\partial\psi^i} \frac{\partial}{\partial\psi^j} \,, \tag{25}$$

where

$$e_3 = \frac{1}{\sin\chi} \left(\frac{\partial}{\partial\eta} - \cos\chi \left[\frac{\partial}{\partial\psi^1} + \frac{\partial}{\partial\psi^2} \right] \right) \,, \tag{26}$$

and

$$\Box_\theta = \frac{1}{h(\theta)b(\theta)^2\sqrt{\det a(\theta)}} \frac{\partial}{\partial\theta} \left[\frac{b(\theta)^2\sqrt{\det a(\theta)}}{h(\theta)} \frac{\partial}{\partial\theta} \right] \,, \tag{27}$$

where a^{ij} is the inverse matrix of a_{ij}.

The eigenvalue equation can be separated by using the left acting $SU(2) \times U(1)$ isometry

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to rewrite the Laplacian in terms of the symmetry generators

$$
\begin{aligned}
\xi_1 &= -\cot\chi\sin\left(\frac{\psi^1+\psi^2}{2}\right)\left[\frac{\partial}{\partial\psi^1}+\frac{\partial}{\partial\psi^2}\right]+\cos\left(\frac{\psi^1+\psi^2}{2}\right)\frac{\partial}{\partial\chi}+\frac{\sin\left(\frac{\psi^1+\psi^2}{2}\right)}{\sin\chi}\frac{\partial}{\partial\eta}\,,\\
\xi_2 &= -\cot\chi\cos\left(\frac{\psi^1+\psi^2}{2}\right)\left[\frac{\partial}{\partial\psi^1}+\frac{\partial}{\partial\psi^2}\right]-\sin\left(\frac{\psi^1+\psi^2}{2}\right)\frac{\partial}{\partial\chi}+\frac{\cos\left(\frac{\psi^1+\psi^2}{2}\right)}{\sin\chi}\frac{\partial}{\partial\eta}\,,\\
\xi_3 &= \frac{\partial}{\partial\psi^1}+\frac{\partial}{\partial\psi^2}\,,\\
\xi_4 &= \frac{\partial}{\partial\psi^1}-\frac{\partial}{\partial\psi^2}\,.
\end{aligned}
\tag{28}
$$

Define the $SU(2)$ quadratic Casimir

$$
\begin{aligned}
\xi^2 &\equiv \xi_1^2+\xi_2^2+\xi_3^2\\
&= (e_3)^2+\frac{1}{\sin\chi}\frac{\partial}{\partial\chi}\sin\chi\frac{\partial}{\partial\chi}+\left[\frac{\partial}{\partial\psi^1}+\frac{\partial}{\partial\psi^2}\right]^2\,.
\end{aligned}
\tag{29}
$$

The Laplacian may thus be written as

$$
\Box=\Box_\theta+\frac{1}{b(\theta)^2}\left[\xi^2-\xi_3^2\right]+\frac{a^{11}}{4}\left[\xi_3+\xi_4\right]^2+\frac{a^{22}}{4}\left[\xi_3-\xi_4\right]^2+2\frac{a^{12}}{4}\left[\xi_3+\xi_4\right]\left[\xi_3-\xi_4\right]\,.
\tag{30}
$$

Now note that ξ^2,ξ_3,ξ_4 all commute with each other. We know their eigenvalues from $SU(2)$ group theory. We must be careful to account for the coordinate ranges correctly. Let ϕ be a simultaneous eigenfunction. Then

$$
\begin{aligned}
\frac{1}{2}\left[\xi_3+\xi_4\right]\phi &= -in_1\phi\,, & n_1 &\in \frac{k_1}{2}\mathbb{Z}\,,\\
\frac{1}{2}\left[\xi_3-\xi_4\right]\phi &= -in_2\phi\,, & n_2 &\in \frac{k_2}{2}\mathbb{Z}\,,\\
\xi^2\phi &= -j(j+1)\phi\,, & j &\in \frac{1}{2}\mathbb{Z}^+\cup\{0\}\,.
\end{aligned}
\tag{31}
$$

The range of n_1+n_2 is restricted by $-j\le n_1+n_2\le j$. All the eigenvalues have a degeneracy of $2[j]+1$, where $[x]$ denotes the integer part of x. The allowed values of n_1+n_2 follow from the standard angular momentum result $n_1+n_2\in\{-j,-j+1,-j+2,\ldots,j\}$ and then retaining only the values for n_1+n_2 that are consistent with (31). We state the allowed values more explicitly below in (42) for the homogeneous case.

One is left with an ordinary differential equation for the θ dependence of the eigenstate

$$
\boxed{-\Box_\theta\phi+\frac{j(j+1)-(n_1+n_2)^2}{b^2}\phi+a^{ij}n_in_j\phi=\lambda\phi\,.}
\tag{32}
$$

Solutions of this equation will introduce another discrete parameter l labelling the eigenvalues. Write the eigenvalues of the full Laplacian as $\lambda_{j,n_1,n_2,l}$. The zeta function for the

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Laplacian on the Einstein manifolds is thus

$$\zeta_{-\square}(s) = \sum_j \sum_{n_1, n_2} \sum_l \frac{2[j] + 1}{\lambda_{j,n_1,n_2,l}^s}, \tag{33}$$

with the summations restricted by (31) and the comments following.

3.1 Homogeneous metrics: spectrum

In the homogeneous case one has $\nu_1 = \nu_2 \equiv \nu$ and

$$k_1 = k_2 = k = \nu \frac{\nu^2 + 2}{2\nu^2 + 1}. \tag{34}$$

The Laplacian simplifies. The relevant terms are collected in appendix A. The equation becomes

$$-\frac{1}{\sin\theta\cos\theta} \frac{\partial}{\partial\theta} \left[\sin\theta\cos\theta \frac{\partial}{\partial\theta}\right] \phi + 4\frac{(2\nu^2 + 1)^2}{(\nu^2 + 2)^2} \left[\frac{n_1^2}{\nu^2 \sin^2\theta} + \frac{n_2^2}{\nu^2 \cos^2\theta}\right] \phi =$$
$$-4\frac{(2\nu^2 + 1)}{\nu^2 + 2} j(j+1)\phi + 6\frac{(2\nu^2 + 1)}{(\nu^2 + 2)^2}(n_1 + n_2)^2 \phi + \frac{(1 + \nu^2)}{\nu^2 + 2}\lambda\phi. \tag{35}$$

This equation may be solved in terms of hypergeometric functions. If one rewrites the equation as

$$-\frac{1}{\sin\theta\cos\theta} \frac{\partial}{\partial\theta} \left[\sin\theta\cos\theta \frac{\partial}{\partial\theta}\right] \phi + \left[\frac{A^2}{\sin^2\theta} + \frac{B^2}{\cos^2\theta}\right] \phi = \Lambda\phi, \tag{36}$$

then two linearly independent solutions are

$$\phi_{\pm} = \cos^{\pm B}\theta \sin^A \theta \, {}_2F_1 \left(\frac{1 + A \pm B - (1 + \Lambda)^{1/2}}{2}, \frac{1 + A \pm B + (1 + \Lambda)^{1/2}}{2}, 1 \pm B; \cos^2\theta\right). \tag{37}$$

Regularity at $\theta = \pi/2$ requires that $\pm B = |B|$. Regularity as $\theta \to 0$ requires both $A = |A| \geq 0$ and further that the hypergeometric function be polynomial. We will see below that the solutions are in fact Jacobi polynomials. This condition for (37) to be a polynomial is that

$$\frac{1 + |A| + |B| \pm (1 + \Lambda)^{1/2}}{2} = -L, \qquad L \in \mathbb{Z}^+ \cup \{0\}. \tag{38}$$

The solution of this equation is $\Lambda = 4l(l + 1)$ with

$$l = L + \frac{2\nu^2 + 1}{\nu(\nu^2 + 2)}(|n_1| + |n_2|). \tag{39}$$

The eigenvalues of the Laplacian are thus seen to be

$$\boxed{\lambda_{j,n_1,n_2,l} = \frac{4(2\nu^2 + 1)}{\nu^2 + 1}j(j+1) + \frac{4(\nu^2 + 2)}{\nu^2 + 1}l(l+1) - \frac{6(2\nu^2 + 1)}{(\nu^2 + 1)(\nu^2 + 2)}(n_1 + n_2)^2.} \tag{40}$$

In the following subsection we clarify the values that n_i, j, l may take. We will recover the spectrum (40) from a semiclassical quantisation of geodesic energies in section 5.2 below.

3.2 Towards the zeta function

To calculate the zeta function later it will be convenient to parameterise the eigenvalues with the following four integers

$$J, L \in \mathbb{Z}^+ \cup \{0\}, \qquad N_1, N_2 \in \mathbb{Z}. \tag{41}$$

The eigenvalues (40) are then calculated using

$$
\begin{aligned}
n_i &= \frac{k}{2} N_i, \\
j &= \frac{k}{2}|N_1 + N_2| + J, \\
l &= \frac{1}{2}\left(|N_1| + |N_2|\right) + L.
\end{aligned}
\tag{42}
$$

The zeta function may be written

$$\zeta_{-\square}(s) = \sum_{L=0}^{\infty}\sum_{J=0}^{\infty}\sum_{N_1=-\infty}^{\infty}\sum_{N_2=-\infty}^{\infty} \frac{2[k|N_1 + N_2|/2] + 2J + 1}{\lambda_{N_1,N_2,J,L}^s}, \tag{43}$$

In the following section we will use this parameterisation to calculate explicitly the zeta function for the homogeneous metrics as $\nu \to \infty$. These are the metrics $T^{p,1}$ with p large.

3.3 Inhomogeneous metrics

The differential equation for the θ dependent part of the solutions (32) is significantly more complicated in the general inhomogeneous case. The first step towards extracting information from the equation is to clarify the structure of the singular points of the equation.

Let us write $z = \cos 2\theta$. Then (32) becomes

$$\frac{d^2}{dz^2}\phi + P(z)\frac{d}{dz}\phi + Q(z)\phi = \omega(z)\lambda\phi, \tag{44}$$

where

$$P(z) = \frac{1}{z-1} + \frac{1}{z+1} + \frac{1}{z-z_0}, \tag{45}$$

and

$$Q(z) - \omega(z)\lambda = \frac{Q_1}{(z-1)^2} + \frac{Q_2}{(z+1)^2} + \frac{Q_3}{(z-z_0)^2} + \frac{R_1}{z-1} + \frac{R_2}{z+1} + \frac{R_3}{z-z_0}, \tag{46}$$

with $z_0 = 2\cos^2\theta_0 - 1$ and

$$\cos^2\theta_0 = \frac{(1-\nu_2^2)(2-\nu_1^2-\nu_1^2\nu_2^2)}{(\nu_1^2-\nu_2^2)(1-\nu_1^2\nu_2^2)}. \tag{47}$$

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The pole z_0 arose as a root of $\Delta_\theta = 0$ in [5]. The coefficients Q_i are given by

$$Q_1 = -\left(\frac{n_1}{k_1}\right)^2, \quad Q_2 = -\left(\frac{n_2}{k_2}\right)^2, \tag{48}$$

and

$$Q_3 = -\frac{1}{(2 - \nu_1^2 - \nu_2^2)(1 - \nu_1^2\nu_2^2)}\left(\frac{\nu_1(1 - \nu_2^2)n_1}{k_1} + \frac{\nu_2(1 - \nu_1^2)n_2}{k_2}\right)^2. \tag{49}$$

The other coefficients R_i satisfy a relation $R_1 + R_2 + R_3 = 0$ since $Q(z) = o(1/z^2)$. The explicit form of these coefficients is somewhat complicated, so we leave them to appendix B. Thus (44) is a Fuchs type differential equation with 4 regular singular points including ∞. Such equations have a canonical form studied by Heun. We transform (44) to the canonical form in appendix E.

From (45) and (48), the eigenfunction around $z = 1$ ($\theta = 0$) takes the form

$$\phi(z) = (1 - z)^{|n_1/k_1|} f(z) \tag{50}$$

with $f(z)$ an analytic function at $z = 1$, while the eigenfunction around $z = -1$ ($\theta = \pi/2$) takes the form

$$\phi(z) = (1 + z)^{|n_2/k_2|} g(z) \tag{51}$$

with $g(z)$ an analytic function at $z = -1$. In the homogeneous case, we saw that the regular function is given by a polynomial.

At infinity, the regular singular point implies that solution behaves as

$$\phi(z) = z^{-1 \pm \sqrt{1 - Q_1 - Q_2 - Q_3 + (z_0 - 1)R_1 + (z_0 + 1)R_2}} h(z) \tag{52}$$

where $h(z)$ is analytic at infinity. For the homogeneous metrics $z_0 \to \infty$, and (52) needs to be recalculated. However, infinity remains a regular singular point in the homogeneous limit.

3.4 Slightly inhomogeneous metrics: spectrum

In this subsection, we compute the Laplacian spectrum for metrics that are slightly inhomogeneous. We can do this using Quantum Mechanical perturbation theory. In the rational limit at least, there are metrics sufficiently near the homogeneous metrics in ν_i space for which perturbation theory is applicable.

The perturbation we consider is

$$\frac{\nu_2}{\nu_1} = 1 + \epsilon, \quad \text{with} \quad \epsilon \ll 1. \tag{53}$$

Equation (44) may be multiplied by $z^2 - 1$ so that it is expressed in terms of a self-adjoint operator H. We will check self-adjointness shortly.

$$H\phi \equiv (z^2 - 1) \left[\frac{d^2}{dz^2}\phi(z) + P(z)\frac{d}{dz}\phi(z) + Q(z)\phi(z) \right] = \tilde{\omega}(z)\lambda\phi(z), \tag{54}$$

with $\tilde{\omega}(z) = (z^2 - 1)\omega(z)$. The coefficients are then perturbed to first order as

$$P = P^{(0)} + \epsilon P^{(1)}, \qquad Q = Q^{(0)} + \epsilon Q^{(1)}, \qquad \tilde{\omega} = \tilde{\omega}^{(0)} + \epsilon\tilde{\omega}^{(1)}. \tag{55}$$

The zeroth order operator is thus

$$H^{(0)} = (z^2 - 1) \left(\frac{d^2}{dz^2} + P^{(0)}(z)\frac{d}{dz} + Q^{(0)}(z) \right). \tag{56}$$

Using the explicit form of the function

$$P^{(0)} = \frac{1}{z - 1} + \frac{1}{z + 1}, \tag{57}$$

the operator becomes

$$H^{(0)} = \frac{d}{dz}(z^2 - 1)\frac{d}{dz} + (z^2 - 1)Q^{(0)}, \tag{58}$$

which is manifestly self-adjoint with respect to the inner product

$$< \phi_1, \phi_2 > = \int_{-1}^{1} dz\phi_1(z)\phi_2(z). \tag{59}$$

Expanding the eigenfunction $\phi = \phi^{(0)} + \epsilon\phi^{(1)}$ and eigenvalue $\lambda = \lambda^{(0)} + \epsilon\lambda^{(1)}$, we have the well known result from first order perturbation theory

$$\lambda^{(1)} = \frac{< \phi^{(0)}, (H^{(1)} - \lambda^{(0)}\tilde{\omega}^{(1)})\phi^{(0)} >}{< \phi^{(0)}, \tilde{\omega}^{(0)}\phi^{(0)} >}. \tag{60}$$

In the present case we have

$$\begin{aligned} H^{(1)} - \lambda^{(0)}\tilde{\omega}^{(1)} &= (z^2 - 1) \left(P^{(1)}\frac{d}{dz} + Q^{(1)} - \lambda^{(0)}\omega^{(1)} \right) \\ &= -\frac{\nu^2(1 + \nu^2)}{(\nu^2 - 1)(\nu^2 + 2)}(z^2 - 1)\frac{d}{dz} \\ &+ Q_1^{(1)}\frac{z + 1}{z - 1} + Q_2^{(1)}\frac{z - 1}{z + 1} + (R_1^{(1)} + R_2^{(1)})z + R_1^{(1)} - R_2^{(1)}, \end{aligned} \tag{61}$$

the actual expressions for $R_i^{(1)}$ and $Q_i^{(1)}$ are still rather large, so we relegate them to appendix B. The denominator in (60) is given by

$$\tilde{\omega}^{(0)} = \frac{1 + \nu^2}{4(2 + \nu^2)}. \tag{62}$$

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In calculating (61) one should keep n_1, n_2, j, l fixed under the perturbation $\nu_2/\nu_1 = 1 + \epsilon$. This is because these are properties of the zeroth order solution, not of the full equation. The relationship between n_i, j, l and N_i, J, L is kept to be that of the homogeneous case (42) and is not modified.

To evaluate (60), note that $\phi^{(0)}(z)$ is just the homogeneous solution that we found previously (37). It is convenient at this point to rewrite these solutions in terms of Jacobi polynomials. It turns out that, up to an overall normalisation

$$\phi^{(0)}(z) = (1-z)^{|N_1|/2}(1+z)^{|N_2|/2}P_L^{(|N_1|,|N_2|)}(z). \tag{63}$$

We have used the integers introduced in (41). We recall that the Jacobi polynomials have the following definition

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(z) &= \frac{(-1)^n}{2^n n!}(1-z)^{-\alpha}(1+z)^{-\beta}\left(\frac{d}{dz}\right)^n [(1-z)^{\alpha+n}(1+z)^{\beta+n}] \\
&\propto F(-n, 1+n+\alpha+\beta, 1+\alpha; (1-z)/2).
\end{aligned} \tag{64}
$$

The advantage of introducing Jacobi polynomials is that they satisfy identities that will enable us to perform the various integrals needed in the evaluation of (60). We see that we will need to evalute $< \phi^{(0)}, z\phi^{(0)} >, < \phi^{(0)}, 1/(z-1)\phi^{(0)} >, < \phi^{(0)}, 1/(z+1)\phi^{(0)} >$ and $< \phi^{(0)}, (z^2-1)d/dz\phi^{(0)} >$. All of these expressions may be computed, the required identities involving Jacobi polynomials are given in appendix C.

The result is

$$
\begin{aligned}
&< \phi^{(0)}, (H^{(1)} - \lambda^{(0)}\tilde{\omega}^{(1)})\phi^{(0)} > \\
&= \frac{\nu^2(1+\nu^2)}{(\nu^2-1)(\nu^2+2)}\left[X_2 + \frac{|N_2|-|N_1|}{2} - X_1\left(L + \frac{|N_1|+|N_2|}{2}\right)\right] \\
&+ (R_1^{(1)} + R_2^{(1)})X_1 + R_1^{(1)} - R_2^{(1)} + Q_1^{(1)} + Q_2^{(1)} \\
&- \frac{|N_1|+|N_2|+1+2L}{|N_1|}Q_1^{(1)} - \frac{|N_1|+|N_2|+1+2L}{|N_2|}Q_2^{(1)},
\end{aligned} \tag{65}
$$

with the coefficients

$$
\begin{aligned}
X_1 &= \frac{N_2^2 - N_1^2}{(2L+|N_1|+|N_2|)(2L+|N_1|+|N_2|+2)}, \\
X_2 &= \frac{L(|N_1|-|N_2|)}{2L+|N_1|+|N_2|}.
\end{aligned} \tag{66}
$$

This expression does not appear to simplify to a pleasant expression in terms of n_i, j, l. We will see in section 5.4 below how a very similar result for the perturbed spectrum may be found from a semiclassical quantisation of the classical geodesic energies.

4 Zeta function for $T^{p,1}$ at large p

We will consider the homogeneous metrics in the limit $\nu \to \infty$, corresponding to $k \to \infty$ also. The advantage of taking this limit is that summations of discrete eigenvalues may be approximated as integrals over continuous parameters. This fact will enable us to compute the zeta function. There are infinitely many metrics in this limit, evenly spaced in ν. See Figure 2 above. Towards the end of this section we will consider one physical application of the zeta function.

The spectrum becomes (40)

$$\lambda = 8j(j+1) + 4l(l+1) - \frac{12}{\nu^2}(n_1 + n_2)^2 \,. \tag{67}$$

One should express the spectrum in term of the integers of (41)

$$
\begin{aligned}
n_i &= \frac{\nu}{4} N_i \,, \\
j &= J + \frac{\nu}{4} |N_1 + N_2| \,, \\
l &= L + \frac{1}{2}(|N_1| + |N_2|) \,.
\end{aligned}
\tag{68}
$$

where $J, L \in \mathbb{Z}^+ \cup \{0\}$ and $N_i \in \mathbb{Z}$. The degeneracy of the eigenvalues is $2[j] + 1$.

In the large ν limit the spectrum may further be simplified to

$$\lambda = 8\left[J + \frac{\nu}{4}|N_1 + N_2|\right]^2 + 4\left[L + \frac{1}{2}(|N_1| + |N_2|)\right]\left[L + \frac{1}{2}(|N_1| + |N_2|) + 1\right] \,. \tag{69}$$

This step does not hold when $N_1 + N_2 = 0$, we will consider this case separately later. Until specified otherwise, assume that $N_1 + N_2 \neq 0$. The degeneracy may now be taken to be $2J + \frac{\nu}{2}|N_1 + N_2|$.

The zeta function at large ν may therefore be written

$$\zeta(s) = \sum_{N_1,N_2,J,L} \frac{\nu^{1-2s}\left(2\frac{J}{\nu} + \frac{1}{2}|N_1 + N_2|\right)}{\left(2\left[2\frac{J}{\nu} + \frac{1}{2}|N_1 + N_2|\right]^2 + \frac{4}{\nu^2}\left[L + \frac{1}{2}(|N_1| + |N_2|)\right]\left[L + \frac{1}{2}(|N_1| + |N_2|) + 1\right]\right)^s} \,. \tag{70}$$

Note that ν is large and J only appears as J/ν. We can approximate this sum by an integral when $N_1 + N_2 \neq 0$. That is to say, we write

$$\sum_{J=0}^{\infty} \to \nu \int_0^\infty d\left(\frac{J}{\nu}\right) \,. \tag{71}$$

The integral is straightforward to perform and the result is

$$\frac{-\nu^{2-2s}}{8(1-s)} \sum_{N_1,N_2,L} \left(\frac{1}{2}(N_1 + N_2)^2 + \frac{4}{\nu^2}\left[L + \frac{1}{2}(|N_1| + |N_2|)\right]\left[L + \frac{1}{2}(|N_1| + |N_2|) + 1\right]\right)^{1-s} \,. \tag{72}$$

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In order to do these sums, the modulus signs force us to consider four cases separately. First note that we can take $N_1 \geq 0$ and the remaining eigenvalues have an extra degeneracy of 2, which we will include at the end. The four cases then depend on the value of N_2. The cases are: (I) $N_2 \geq 0$. (II) $N_2 = -N_1$. (III) $-\infty < N_2 < -N_1$. (IV) $-N_1 < N_2 < 0$. The total zeta function will be the sum of these contributions

$$\zeta(s) = 2\left[\zeta_I(s) + \zeta_{II}(s) + \zeta_{III}(s) + \zeta_{IV}(s)\right] . \tag{73}$$

A check of our result is given in appendix D.

4.1   Case I: $N_2 \geq 0$

In this case we may write $M = N_1 + N_2 > 0$. We consider the case $M = 0$ later. The sum is expressed entirely in terms of M. This introduces a degeneracy of $M + 1$: the number of ways of writing M as the sum of two positive integers. The zeta function becomes

$$\zeta_I(s) = \frac{\nu^{2-2s}}{8(s-1)} \sum_{M,L} \frac{M+1}{\left(\frac{M^2}{2} + 4(\frac{L+M/2}{\nu})(\frac{L+M/2+1}{\nu})\right)^{s-1}} . \tag{74}$$

We may now convert the L summation into an integral for the same reasons as before. One obtains

$$\zeta_I(s) = \frac{\nu^{3-2s}}{8(s-1)} \sum_{M=1}^{\infty} \int_0^{\infty} \frac{M+1}{\left(\frac{M^2}{2} + 4(\frac{L+M/2}{\nu})(\frac{L+M/2+1}{\nu})\right)^{s-1}} d\left(\frac{L}{\nu}\right) . \tag{75}$$

This integral can be done using, for example, the Maple program. The answer may be expressed in terms of hypergeometric functions. Taking the large ν limit, one obtains

$$\sum_{M=1}^{\infty} \frac{2^{s+1/2}\pi^{1/2}}{128} \nu^{3-2s} \frac{\Gamma(s-3/2)}{\Gamma(s)} M^{3-2s}(1+M) . \tag{76}$$

But now the remaining sum is just a Riemann zeta function, so the final answer for this case is

$$\zeta_I(s) = \frac{2^{s+1/2}\pi^{1/2}}{128} \nu^{3-2s} \frac{\Gamma(s-3/2)}{\Gamma(s)} \left[\zeta_R(2s-3) + \zeta_R(2s-4)\right] . \tag{77}$$

4.2   Case II: $N_2 = -N_1$

In doing this case we will include the $N_1 = N_2 = 0$ case. In these cases, the integral method is not valid. The zeta function contribution from these values is

$$\zeta_{II}(s) = \sum_{N_1,J,L} \frac{(2J+1)}{(8J(J+1) + 4(L+N_1)(L+N_1+1))^s} + \mathcal{O}(1/\nu^2) . \tag{78}$$

As usual, the summation should exclude the zero mode. This sum may be simplified by considering $Q = L + N_1$. It may be possible to do this sum using contour integration. This will not be necessary here; the only point that is relevant is that to leading order there is no ν dependence and that the first correction is order $1/\nu^2$. This term will not give the dominant contribution to physical quantities which will be at e.g. $s = 0, -1/2$.

4.3 Case III: $-\infty < N_2 < -N_1$

Define the quantities $M = N_1 + N_2 < 0$ and $N = N_1 - N_2 > 0$. The zeta function contribution for this case is

$$\zeta_{III}(s) = \frac{\nu^{2-2s}}{8(s-1)} \sum_{N,M,L} \left[\frac{M^2}{2} + \frac{4}{\nu^2}(L + N/2)(L + N/2 + 1) \right]^{1-s}. \tag{79}$$

What is the range of the summation for M and N? If we fix M, then we see that N takes the values $N = |M|, |M|+2, |M|+4, \ldots$. Therefore $L+N/2$ takes the values $|M|/2, |M|/2+1, |M|/2+2, \ldots$ with degeneracy $L + N/2 + 1 - |M|/2$. Defining $x = (L + N/2)/\nu$ we may rewrite the summation over L and N as an integral over x:

$$\zeta_{III}(s) = \frac{\nu^{4-2s}}{8(s-1)} \sum_{M=-1}^{-\infty} \int_{|M|/2\nu}^{\infty} \frac{x + 1/\nu - |M|/2\nu}{\left(\frac{M^2}{2} + 4x(x + 1/\nu) \right)^{s-1}} dx. \tag{80}$$

One may again do this integral. It is similar to the one considered previously. The final result for the zeta function is

$$\zeta_{III}(s) = \frac{-2^s \, \nu^{4-2s} \zeta_R(2s-4) \left[(s-1)_2 F_1(2, s, 5-s; -1) + (s-4)_2 F_1(1, s-1, 4-s; -1) \right]}{(s-1)(s-2)(s-3)(s-4)}. \tag{81}$$

Note that this expression always has one power more in ν than the contribution from case I (77). Therefore the contribution from case I is negligible unless the expression (81) vanishes. But if this expression were to vanish then we would need to calculate the subleading term. This will not be a problem in practice.

4.4 Case IV: $-N_1 < N_2 < 0$

Define the quantities $M = N_1 + N_2 > 0$ and $N = N_1 - N_2 > 0$. The zeta function contribution for this case is

$$\zeta_{IV}(s) = \frac{\nu^{2-2s}}{8(s-1)} \sum_{N,M,L} \left[\frac{M^2}{2} + \frac{4}{\nu^2}(L + N/2)(L + N/2 + 1) \right]^{1-s}. \tag{82}$$

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This is the same expression as before (79) except that now if we fix M the range of N is $M + 2, M + 4, ...$ Now $L + N/2$ takes the values $M/2 + 1, M/2 + 2, ...,$ with degeneracy $L + N/2 - M/2$. Defining $x = (L + N/2)/\nu$ as before the zeta function becomes

$$\zeta_{IV}(s) = \frac{\nu^{4-2s}}{8(s-1)} \sum_{M=1}^{\infty} \int_{M/2\nu+1/\nu}^{\infty} \frac{x - M/2\nu}{\left(\frac{M^2}{2} + 4x(x + 1/\nu)\right)^{s-1}} dx \, . \tag{83}$$

To leading order in ν this integral turns out to be the same as in the previous case

$$\zeta_{IV}(s) = \zeta_{III}(s) \, . \tag{84}$$

4.5   Physical applications of the zeta function

The most immediate application of the zeta function is to calculate the thermodynamics of a free scalar field on the compact manifold M. We can add a mass or coupling to the Ricci scalar without changing the zeta function that we have calculated to leading order in ν, as can be seen from (69). We denote this coupling with κ. More precisely, κ will only be relevant when $|N_1 + N_2| = 0$, and we saw that this case gives subleading contributions to the zeta function.

The free energy is given by the logarithm of the partition function on $M \times S^1$, where the S^1 has length β and the temperature of the scalar radiation is $T = 1/\beta$ as usual,

$$
\begin{aligned}
F &= -\frac{1}{\beta} \ln Z = -\frac{1}{\beta} \ln \int \mathcal{D}\phi \, e^{-\int_0^\beta dt \int d^d x \sqrt{g} \left[\frac{1}{2}(\nabla\phi)^2 + \frac{\kappa}{2}\phi^2\right]} \\
&= \frac{1}{2\beta} \ln \det \frac{-(\partial/\partial t)^2 - \Box + \kappa}{\mu^2} \, ,
\end{aligned} \tag{85}
$$

where we have introduced an arbitrary mass scale μ so that the dimensionalities are correct. Zeta function regularisation [15, 16] then gives a finite expression for the formal determinant

$$F = -\frac{1}{2\beta} \zeta_A'(0) \, , \tag{86}$$

where we have set $A = [-(\partial/\partial t)^2 - \Box + \kappa]/\mu^2$. The zeta function is analytic at the origin, so the free energy is finite.

We can calculate the free energy in the limits of low and high temperature. In both cases, it is convenient to use the following expression for the zeta function

$$\zeta_A(s) = \mathrm{Tr} A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \mathrm{Tr} e^{-\tau A} \, . \tag{87}$$

The eigenvalues of $(\partial/\partial t)^2$ on the S^1 factor are

$$\omega_n^2 = \left(\frac{2\pi}{\beta}\right)^2 n^2 \, , \quad n \in \mathbb{Z} \, . \tag{88}$$

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It follows that we may write the zeta function as

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\tau \tau^{s-1} e^{-\tau\left(\frac{2\pi}{\mu\beta}\right)^2 n^2} \text{Tr}_d e^{-\tau A_S} , \tag{89}$$

where A_S refers to the operator acting on the d dimensional spatial section M.

First consider the theory at low temperature. In this regime we may approximate the sum over the S^1 modes by an integral

$$\sum_{n=-\infty}^{\infty} e^{-\tau\left(\frac{2\pi}{\mu\beta}\right)^2 n^2} \rightarrow \frac{\beta\mu}{2\pi} \int_{-\infty}^{\infty} dk e^{-\tau k^2} = \frac{\beta\mu}{(4\pi)^{1/2}} \frac{1}{\tau^{1/2}} . \tag{90}$$

The zeta function may thus be written as

$$
\begin{aligned}
\zeta_A(s)|_{\text{Low T}} &= \frac{\beta\mu}{\Gamma(s)(4\pi)^{1/2}} \int_0^{\infty} d\tau \tau^{s-1/2-1} \text{Tr}_d e^{-\tau A_S} , \\
&= \frac{\beta\mu}{(4\pi)^{1/2}} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta_{A_S}(s-1/2) .
\end{aligned} \tag{91}
$$

The technique for calculating the high temperature behaviour was introduced by [17]. For the terms in the sum with $n \neq 0$, it turns out that we can use the Schwinger-de Witt expansion for the heat kernel $\text{Tr}_d e^{-\tau A_S}$ as $\tau \rightarrow 0$. This will work when the temperature is larger than the curvatures, because in this case the dominant contribution to the integral comes from small values of τ. The Schwinger-de Witt asymptotic expansion is

$$\text{Tr}_d e^{-\tau A_S} \sim \frac{\mu^d}{(4\pi)^{d/2}} \sum_{k=0}^{\infty} \frac{a_{2k}}{\mu^{2k}} \tau^{k-d/2} = \frac{\mu^d}{(4\pi)^{d/2}} \left[a_0 \tau^{-d/2} + \frac{a_2}{\mu^2} \tau^{1-d/2} + \cdots \right] , \tag{92}$$

where the first few a_{2k} are given in appendix D. If we let $\mathfrak{R}^k$ denote a generic curvature scalar with mass dimension $2k$, then we have that $a_{2k} = \mathcal{O}(\mathfrak{R}^k)$.

Substituting the expansion (92) into the expression for the zeta function (89) and doing the integral over τ, one obtains

$$
\begin{aligned}
\zeta_A(s)|_{\text{High T}} &= \zeta_{A_S}(s) + \frac{2\mu^d}{(4\pi)^{d/2}\Gamma(s)} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{4\pi^2 n^2}{\mu^2\beta^2}\right)^{d/2-s-k} \Gamma(s+k-d/2) \frac{a_{2k}}{\mu^{2k}} \\
&= \zeta_{A_S}(s) + \frac{2\mu^d}{(4\pi)^{d/2}\Gamma(s)} \sum_{k=0}^{\infty} \left(\frac{4\pi^2}{\mu^2\beta^2}\right)^{d/2-s-k} \Gamma(s+k-d/2)\zeta_R(2s+2k-d) \frac{a_{2k}}{\mu^{2k}}
\end{aligned} \tag{93}
$$

where ζ_R is the Riemann zeta function. We see that the series expansion will be valid if $|a_{2k}\beta^{2k}| \ll 1$, that is, if the temperature is large compared with curvature scales $|\mathfrak{R}| \ll T^2$.

Thermodynamic quantities may be calculated using standard formulae such as $E = \partial(\beta F)/\partial\beta$ and $S = -\partial F/\partial T$. The results in the low and high temperature limits are shown in Table 2. In this table we restore the dependence on the scalar curvature R for completeness. At low temperature, the leading contribution comes from $\zeta_{III}(s)$ and $\zeta_{IV}(s)$.

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Quantity	Low Temperature	High Temperature
f	$-\dfrac{\sqrt{10R}\nu^5}{2419200}+\cdots$	$-\dfrac{2\sqrt{2}\pi^6}{945\nu}T^6+\dfrac{(6\kappa+R)\sqrt{2}\pi^4}{2160\nu}T^4+\cdots$
E	$-\dfrac{\sqrt{10R}\,\nu^5}{2419200}+\cdots$	$\dfrac{2\sqrt{2}\pi^6}{189\nu}T^6-\dfrac{(6\kappa+R)\sqrt{2}\pi^4}{720\nu}T^4\cdots$
S	$0+\cdots$	$\dfrac{4\sqrt{2}\pi^6}{315\nu}T^5-\dfrac{(6\kappa+R)\sqrt{2}\pi^4}{540\nu}T^3\cdots$

Table 2: Thermodynamics of a scalar field on $M\times S^1$ at high and low temperature.

5   Hamilton-Jacobi equation and geodesics

5.1   Separability of the Hamilton-Jacobi equation

In this section we show that the Hamilton-Jacobi equation for the geodesics of the metrics we are considering is integrable. Hence we obtain first order equations for the geodesics.

The Lagrangian for a free particle is

$$\mathcal{L}=g_{ab}dx^a dx^b=h^2\dot{\theta}^2+b^2(\dot{\chi}^2+\sin^2\chi\,\dot{\eta}^2)+a_{ij}(\dot{\psi}^i+\cos\chi\,\dot{\eta})(\dot{\psi}^j+\cos\chi\,\dot{\eta})\,, \quad (94)$$

where a dot denotes differentiation with respect to some time parameterisation of the geodesics. The geodesic equations may be separated by using a Hamilton-Jacobi method. In this description, the dual momenta are

$$p_a=\frac{\partial\mathcal{L}}{\partial\dot{x}^a}=\frac{\partial S}{\partial x^a}\,. \quad (95)$$

The coordinates ψ^i and η are cyclic, giving three conserved momenta, $p_{\psi^i}=J_i$ and $p_\eta=J_3$. The first order equations for these coordinates are easily seen to be

$$\dot{\eta}=\frac{1}{2}\frac{(J_1+J_2)\cos\chi-J_3}{b^2\sin^2\chi}\,,$$
$$\dot{\psi}^i=\frac{1}{2}a^{ij}J_j-\cos\chi\,\dot{\eta}\,. \quad (96)$$

To find the remaining geodesic equations, consider the Hamilton-Jacobi equation

$$\frac{1}{h(\theta)^2}\left(\frac{\partial S}{\partial\theta}\right)^2+\frac{1}{b(\theta)^2}\left(\frac{\partial S}{\partial\chi}\right)^2+\frac{1}{b(\theta)^2}\,(e_3 S)^2+a^{ij}\frac{\partial S}{\partial\psi^i}\frac{\partial S}{\partial\psi^j}=E\,, \quad (97)$$

with E a constant. This equation may be separated by taking

$$S=W(\theta)+U(\chi)+J_3\eta+J_1\psi^1+J_2\psi^2\,. \quad (98)$$

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One then obtains the equations

$$\left(\frac{dU}{d\chi}\right)^2 + \frac{1}{\sin^2\chi}\left(J_3 - \cos\chi[J_1 + J_2]\right)^2 = L^2,\tag{99}$$

and

$$\frac{1}{h^2}\left(\frac{dW}{d\theta}\right)^2 + a^{ij}J_iJ_j + \frac{L^2}{b^2} = E,\tag{100}$$

where L is a constant. Now use the relations

$$\begin{aligned}
\frac{dU}{d\chi} &= \frac{\partial S}{\partial \chi} = \frac{\partial \mathcal{L}}{\partial \dot\chi} = 2b^2\dot\chi,\\
\frac{dW}{d\theta} &= \frac{\partial S}{\partial \theta} = \frac{\partial \mathcal{L}}{\partial \dot\theta} = 2h^2\dot\theta,
\end{aligned}\tag{101}$$

to obtain equations for $\dot\chi$ and $\dot\theta$ from (99) and (100) respectively. These equations can then be used to eliminate the time dependence and hence obtain equations for the orbits. For example

$$\frac{h^2}{b^2}\left(\frac{d\theta}{d\chi}\right)^2 = \frac{E^2 - L^2/b^2 - a^{ij}J_iJ_j}{L^2\sin^2\chi - [J_3 - \cos\chi(J_1 + J_2)]^2}\sin^2\chi.\tag{102}$$

Similar equations may be derived for ψ^i and η in terms of χ. In fact, one obtains a simple equation for $d\chi/d\eta$

$$\left(\frac{d\chi}{d\eta}\right)^2 = \sin^2\chi\left(\frac{L^2\sin^2\chi}{(J_3 - \cos\chi(J_1 + J_2))^2} - 1\right).\tag{103}$$

This equation describes the projection of the geodesic onto the (χ, η) two-sphere. One can recognise (103) as describing the motion of a charged particle on a sphere with a magnetic monopole background. This is not surprising given that the full metric was constructed from a principle T^2 bundle over the S^2. It is well known that charged particle motion on a sphere with a magnetic monopole background results in closed circular orbits. A little algebra shows that the motion (103) describes a circle on the two sphere about an axis $\mathbf{n}$ with opening angle Θ. One finds

$$\sin^2\Theta = \frac{L^2}{(J_1 + J_2)^2 + L^2},\tag{104}$$

and

$$\mathbf{n} = \sqrt{1 - \frac{J_3^2}{L^2}\sin^2\Theta}\left[\sin\Xi\,\mathbf{e}_x + \cos\Xi\,\mathbf{e}_y\right] + \frac{J_3}{L}\sin\Theta\mathbf{e}_z,\tag{105}$$

where Ξ is an arbitrary angle and $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is the standard Cartesian basis.

5.2 Action-angle analysis of the homogeneous case: semiclassical spectrum

In this subsection we carry out a full action-angle variable analysis of the geodesics of the homogeneous metrics. This analysis allows us to compute the frequencies of closed geodesics and also allows a semiclassical quantisation of the system. We see that the semicalssical spectrum essentially agrees with the spectrum of the Laplacian that we calculated previously.

We work in phase space

$$M = T^*(S^2 \times S^3)\,, \tag{106}$$

with coordinates $(x^a) = (\beta = 2\theta, \chi, \eta, \psi_1, \psi_2)$ and their dual momenta p_a. The canonical one form α and corresponding symplectic form ω are as always

$$\alpha = p_a \wedge dx^a\,, \qquad \omega = d\alpha\,. \tag{107}$$

The Hamiltonian for the system is

$$H(x,p) = g^{ab} p_a p_b\,. \tag{108}$$

We may write H explicitly as

$$
\begin{aligned}
H &= \frac{4(2+\nu^2)}{1+\nu^2}\left(p_\beta^2 + \frac{1}{k^2 \sin^2\beta}((p_{\psi_2}+p_{\psi_1})^2 - 2(p_{\psi_2}+p_{\psi_1})(p_{\psi_2}-p_{\psi_1})\cos\beta + (p_{\psi_2}-p_{\psi_1})^2)\right) \\
&+ \frac{4(1+2\nu^2)}{1+\nu^2}\left(p_\chi^2 + \frac{1}{\sin^2\chi}(p_\eta^2 - 2(p_{\psi_2}+p_{\psi_1})p_\eta \cos\chi + (p_{\psi_2}+p_{\psi_1})^2)\right) \\
&- \frac{6(1+2\nu^2)}{(1+\nu^2)(2+\nu^2)}(p_{\psi_2}+p_{\psi_1})^2\,.
\end{aligned}
\tag{109}
$$

This Hamiltonian system is integrable since there exist $5 = \dim M/2$ independent functions F_a $(a = 1 \ldots 5)$ such that

- $\{F_a, H\}_\omega = 0$,

- $\{F_a, F_b\}_\omega = 0$.

The functions may be taken to be

$$
\begin{aligned}
F_1^2 &= p_\beta^2 + \frac{1}{k^2 \sin^2\beta}((p_{\psi_2}+p_{\psi_1})^2 - 2(p_{\psi_2}+p_{\psi_1})(p_{\psi_2}-p_{\psi_1})\cos\beta + (p_{\psi_2}-p_{\psi_1})^2) \\
F_2^2 &= p_\chi^2 + \frac{1}{\sin^2\chi}(p_\eta^2 - 2(p_{\psi_2}+p_{\psi_1})p_\eta \cos\chi + (p_{\psi_2}+p_{\psi_1})^2) \\
F_3 &= p_\eta, \qquad F_4 = p_{\psi_1}, \qquad F_5 = p_{\psi_2}.
\end{aligned}
\tag{110}
$$

In which case the Hamiltonian (109) may be written

$$H = \frac{4(2+\nu^2)}{1+\nu^2}F_1^2 + \frac{4(1+2\nu^2)}{1+\nu^2}F_2^2 - \frac{6(1+2\nu^2)}{(1+\nu^2)(2+\nu^2)}(F_4 + F_5)^2 \,. \tag{111}$$

In fact, the quadratic conserved quantities F_1 and F_2 are related to two reducible Staeckel-Killing tensors, K_i^{ab}, of the background

$$F_i = K_i^{ab}p_a p_b \qquad\qquad (i = 1, 2) \,. \tag{112}$$

This follows from writing the quantities as

$$F_1^2 = \tilde{\xi}_1^2 + \tilde{\xi}_2^2 + \tilde{\xi}_3^2 \,, \quad F_2^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \,, \tag{113}$$

where we have used the generators of the $SU(2) \times SU(2)$ symmetry,

$$\begin{aligned}
\xi_1 &= -\cot\chi\sin((\psi_1+\psi_2)/2)(p_{\psi_1}+p_{\psi_2}) + \cos((\psi_1+\psi_2)/2)p_\chi + \frac{\sin((\psi_1+\psi_2)/2)}{\sin\chi}p_\eta \,, \\
\xi_2 &= -\cot\chi\cos((\psi_1+\psi_2)/2)(p_{\psi_1}+p_{\psi_2}) - \sin((\psi_1+\psi_2)/2)p_\chi + \frac{\cos((\psi_1+\psi_2)/2)}{\sin\chi}p_\eta \,, \\
\xi_3 &= p_{\psi_1} + p_{\psi_2} \,,
\end{aligned} \tag{114}$$

and

$$\begin{aligned}
\tilde{\xi}_1 &= -\cot\beta\sin(k(\psi_1+\psi_2)/2)(p_{\psi_1}+p_{\psi_2})/k + \cos(k(\psi_1+\psi_2)/2)p_\beta \\
&\quad - \frac{\sin(k(\psi_1+\psi_2)/2)}{\sin\beta}(p_{\psi_1}-p_{\psi_2})/k \,, \\
\tilde{\xi}_2 &= -\cot\alpha\cos(k(\psi_1+\psi_2)/2)(p_{\psi_1}+p_{\psi_2})/k - \sin(k(\psi_1+\psi_2)/2)p_\beta \\
&\quad - \frac{\cos(k(\psi_1+\psi_2)/2)}{\sin\beta}(p_{\psi_1}-p_{\psi_2})/k \,, \\
\tilde{\xi}_3 &= (p_{\psi_1}+p_{\psi_2})/k \,,
\end{aligned} \tag{115}$$

which satisfy the relations $\{\xi_i, \xi_j\} = -\epsilon_{ijk}\xi_k$ and $\{\tilde{\xi}_i, \tilde{\xi}_j\} = -\epsilon_{ijk}\tilde{\xi}_k$.

The five constants of motion allow us to consider the level set

$$M_f = \{(x,p) : F_a(x,p) = f_a\}. \tag{116}$$

General theorems show that M_f is diffeomorphic to the 5 dimensional torus. The action variables are constructed by considering 5 independent cycles in the torus, C_a, and writing

$$I_a = \frac{1}{2\pi}\oint_{C_a}\alpha\,. \tag{117}$$

The definition is in fact independent of the cycle chosen to represent a given homology class because the symplectic form $\omega = d\alpha$ vanishes when restricted to M_f. From their

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definition, the $\{I_a\}$ variables will be invertible functions of the constants $\{f_a\}$ only. If we invert this relationship, we may consider $f_a(I)$. The key step in action-angle analysis is then to note that the Hamilton-Jacobi function may be considered as a function of x^a and I_a: $S = S(x, I)$. Thus one may define new coordinates

$$\phi^a = \frac{\partial S}{\partial I_a} \, . \tag{118}$$

These are the angle variables. The action-angle variables satisfy three important properties. Firstly, (ϕ^a, I_a) are canonical coordinates on the phase space

$$\omega = d\phi^a \wedge dI_a \, . \tag{119}$$

Secondly, the variables ϕ^a are indeed angles on the cycles C_a

$$\oint_{C_a} d\phi^b = 2\pi \delta_a^b \, . \tag{120}$$

Thirdly, the time evolution equations for (ϕ^a, I_a) are trivial

$$\begin{aligned}
\dot{I}_a &= 0 \, , \\
\dot{\phi}^a &= \frac{\partial H(I)}{\partial I_a} \equiv \Omega^a(I) \Rightarrow \phi^a = \Omega^a(I)t + \text{const.}
\end{aligned} \tag{121}$$

The next step is to choose 5 cycles in M_f. Three cases are particularly straightforward. Consider cycles that have tangent vectors $\frac{\partial}{\partial \eta}$, $\frac{\partial}{\partial \psi_1}$ and $\frac{\partial}{\partial \psi_2}$. The action variables are respectively, using (110),

$$\boxed{I_3 = f_3, \quad I_i = \frac{2}{k} f_i \quad (i = 4, 5).} \tag{122}$$

The remaining two integrals are slightly more complicated. They are most easily computed by re-expressing the action variable as an integral over a surface in phase space with $C_a = \partial S_a$

$$2\pi I_a = \oint_{C_a} \alpha = \int\int_{S_a} d\alpha = \int\int_{S_a} dp_b \wedge dx^b \, . \tag{123}$$

First we describe the curves. The two cycles will be taken to be the intersection of M_f with the surface generated by tangent vectors $(\frac{\partial}{\partial \beta}, \frac{\partial}{\partial p_\beta})$ and $(\frac{\partial}{\partial \chi}, \frac{\partial}{\partial p_\chi})$ respectively. From (110) we can calculate the equation for the cycles, in both cases the curve takes the general form

$$C = \{(x, y) \in [0, \pi] \times \mathbb{R} \,|\, y^2 = a^2 - (b^2 - 2bc \cos x + c^2)/\sin^2 x\} \, . \tag{124}$$

If the following two conditions hold

$$a^2 - |bc| > 0, \qquad (a^2 - b^2)(a^2 - c^2) > 0 \, , \tag{125}$$

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then the equation for the curve becomes

$$y^2 = a^2(\cos x - \delta_1)(\delta_2 - \cos x)/\sin^2 x, \quad -1 \le \delta_1 < \delta_2 \le 1, \tag{126}$$

with

$$\delta_{1,2} = \frac{1}{a^2}\left(bc \pm \sqrt{(a^2 - b^2)(a^2 - c^2)}\right). \tag{127}$$

The curve C is closed and the area A enclosed by C is

$$\begin{aligned} A &= 2|a| \int_{\delta_1}^{\delta_2} \frac{\sqrt{(t - \delta_1)(\delta_2 - t)}}{1 - t^2} dt \\ &= |a|\pi \left(2 - \sqrt{(1 + \delta_1)(1 + \delta_2)} - \sqrt{(1 - \delta_1)(1 - \delta_2)}\right) \\ &= \pi(2|a| - |b + c| - |b - c|). \end{aligned} \tag{128}$$

By (123) we see that the action variables will be given by the area inside the curve in the (x, p) plane. Now consider the two cases:

(A) $I_1(f)$: In the equation for the curve (124) we have $a = f_1, b = (f_4 + f_5)/k, c = (f_5 - f_4)/k$. We can take $f_1 > 0$ without loss of generality. Then, $|I_1(f)| = A_1/2\pi$, where A_1 is the area given by (128)

$$\boxed{|I_1| = f_1 - \left|\frac{f_4}{k}\right| - \left|\frac{f_5}{k}\right|.} \tag{129}$$

(B) $I_2(f)$: Here we have $a = f_2, b = f_4 + f_5, c = f_3$, and taken $f_2 > 0$ without loss of generality. Then, $|I_2(f)| = A_2/2\pi$, where A_2 is the area given by (128):

$$\boxed{|I_2| = f_2 - \frac{1}{2}(|f_4 + f_5 + f_3| + |f_4 + f_5 - f_3|).} \tag{130}$$

Here, we have assumed the condition (125) for f_a, which is equivalent to that of the existence of periodic orbits in M.

We can now calculate the frequencies of the orbits. The frequency Ω^a is (121)

$$\Omega^a = \frac{\partial H(I)}{\partial I_a}, \tag{131}$$

where

$$H(I) = \frac{4(2 + \nu^2)}{1 + \nu^2} f_1^2(I) + \frac{4(1 + 2\nu^2)}{1 + \nu^2} f_2^2(I) - \frac{3\nu^2(\nu^2 + 2)}{2(1 + \nu^2)(1 + 2\nu^2)}(I_4 + I_5)^2. \tag{132}$$

Using (129)and (130) we have

$$\begin{aligned} f_1 &= |I_1| + \frac{1}{2}(|I_4| + |I_5|), \\ f_2 &= |I_2| + \frac{1}{4}(|k(I_4 + I_5) + 2I_3| + |k(I_4 + I_5) - 2I_3|). \end{aligned} \tag{133}$$

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This is a classical version of the Laplace spectrum (40) written with the action coordinates.

We may use the expression for the Hamiltonian in terms of the action variables (132) to perform a semiclassical quantisation of the spectrum. The semiclassical prescription is to replace action variables by integers. Consider the following quantisation

$$I_1 \to L\,, \qquad I_3 \to M\,, \qquad I_4 \to N_1\,, \qquad I_5 \to N_2\,,$$

$$I_2 \to J + \frac{k}{2}|N_1 + N_2| - \frac{1}{4}(|k(N_1 + N_2) + 2M| + |k(N_1 + N_2) - 2M|)\,. \tag{134}$$

The slightly awkward quantisation of I_2 is in order to make contact with our previous notation. It is not difficult to see that the above expression sets I_2 to be an integer and J to be a positive integer.

Using the definitions of (42) we see that the semiclassical spectrum may be written as

$$H = \frac{4(2\nu^2 + 1)}{\nu^2 + 1}j^2 + \frac{4(\nu^2 + 2)}{\nu^2 + 1}l^2 - \frac{6(2\nu^2 + 1)}{(\nu^2 + 1)(\nu^2 + 2)}(n_1 + n_2)^2\,. \tag{135}$$

This is precisely the same as the spectrum for the Laplacian that we calculated previously in (40), up to the usual semiclassical error $j^2 \to j(j+1)$, $l^2 \to l(l+1)$. This agreement suggests an approach for calculating the Laplacian spectrum in the inhomogeneous cases. We can understand the semiclassical disagreement by comparing a naive quantisation of the classical Hamiltonian (109), $p_a \to i\partial_{x^a}$, with the Laplacian (25). The quantised Hamiltonian has terms ∂_β^2 and ∂_χ^2 whilst the Laplacian has $1/\sin\beta\,\partial_\beta[\sin\beta\partial_\beta]$ and $1/\sin\chi\,\partial_\chi[\sin\chi\partial_\chi]$. This represents an ordering ambiguity in quantisation and explains j^2 versus $j(j+1)$.

Finally, we can write down an explicit solution for the functions $W(\beta, I)$ and $U(\chi, I)$ that appeared in the separation of the Hamilton-Jacobi equation (98). Using the following formula, which is the integral of (128) in an indefinite form,

$$
\begin{aligned}
K(t; \delta_1, \delta_2) \;\equiv\; & \int \frac{\sqrt{(t - \delta_1)(\delta_2 - t)}}{1 - t^2}\,dt \\
=\; & \sqrt{(1 + \delta_1)(1 + \delta_2)}\,\arctan\left(\sqrt{\frac{(1 + \delta_1)(\delta_2 - t)}{(1 + \delta_2)(t - \delta_1)}}\right) \\
+\; & \sqrt{(1 - \delta_1)(1 - \delta_2)}\,\arctan\left(\sqrt{\frac{(1 - \delta_1)(\delta_2 - t)}{(1 - \delta_2)(t - \delta_1)}}\right) \\
-\; & 2\arctan\left(\sqrt{\frac{\delta_2 - t}{t - \delta_1}}\right) \quad (-1 \le \delta_1 < \delta_2 \le 1),
\end{aligned}
$$

we find that

$$W(\beta, I) = f_1(I)K(\cos\beta; \delta_1(I), \delta_2(I)),$$

$$U(\chi, I) = f_2(I)K(\cos\chi; \tilde{\delta}_1(I), \tilde{\delta}_2(I)), \tag{136}$$

where the parameters $\delta_i(I)$ and $\tilde{\delta}_j(I)$ are determined as a function of I_a by (127) with the values for a, b, c given in (A) and (B) above, respectively. W and U are multivalued functions which have multiplicities NA_1, NA_2, with $N \in \mathbb{Z}$.

5.3 Action-angle analysis of the inhomogeneous case

After setting $z = \cos 2\theta$, hence $p_\theta = -2p_z\sqrt{1-z^2}$, the Hamiltonian is given by

$$
\begin{aligned}
H & = \frac{4(1-z^2)}{h^2(z)}p_z^2 + \frac{1}{b^2(z)}\left(p_\chi^2 + \frac{1}{\sin^2\chi}(p_\eta^2 - 2(p_{\psi_2} + p_{\psi_1})p_\eta \cos\chi + (p_{\psi_2} + p_{\psi_1})^2 \cos^2\chi)\right) \\
& + a^{ij}(z)p_{\psi_i}p_{\psi_j}\,.
\end{aligned}
\tag{137}
$$

This is also an integrable Hamiltonian system. The following functions are mutually commuting conserved quantities,

$$
\begin{aligned}
F_1 & = H, \\
F_2^2 & = p_\chi^2 + \frac{1}{\sin^2\chi}(p_\eta^2 - 2(p_{\psi_2} + p_{\psi_1})p_\eta \cos\chi + (p_{\psi_2} + p_{\psi_1})^2), \\
F_3 & = p_\eta, \qquad F_4 = p_{\psi_1}, \qquad F_5 = p_{\psi_2}\,.
\end{aligned}
\tag{138}
$$

As in the homogeneous case, the new momentum coordinates are introduced by

$$
I_a(f) = \frac{1}{2\pi}\int_{C_a}\alpha\,.
\tag{139}
$$

Four of the action variables are the same as in the homogeneous case

$$
|I_2| = f_2 - \frac{1}{2}(|f_4 + f_5 + f_3| + |f_4 + f_5 - f_3|)\,,
\tag{140}
$$

and

$$
I_3 = f_3\,, \quad I_4 = \frac{2}{k_1}f_4\,, \quad I_5 = \frac{2}{k_2}f_5\,.
\tag{141}
$$

The coordinate I_1 is harder to calculate. On the level set

$$
M_f = \{(x, p) : F_a(x, p) = f_a\}\,,
\tag{142}
$$

the z-component p_z is written as

$$
p_z^2 = \frac{h(z)^2}{4(1-z^2)}\left(f_1 - \frac{f_2^2 - (f_4 + f_5)^2}{b(z)^2} - a^{ij}(z)f_i f_j\right) \equiv \hat{Q}(z)\,,
\tag{143}
$$

where

$$
\hat{Q}(z) = \frac{Q_1}{(z-1)^2} + \frac{Q_2}{(z+1)^2} + \frac{Q_3}{(z-z_0)^2} + \frac{R_1}{z-1} + \frac{R_2}{z+1} + \frac{R_3}{z-z_0}\,,
\tag{144}
$$

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and the coefficients Q_i, R_j are given by equations (48) and (49) and the equations in appendix B, together with the replacements

$$j(j+1) \to f_2^2, \quad n_1, n_2 \to f_4, f_5, \quad \lambda \to f_1. \tag{145}$$

Given the expression (143) for p_z, we would like to calculate the action variable

$$I_1 = \frac{1}{2\pi} \oint p_z dz. \tag{146}$$

Let us introduce

$$\varphi(z) = (z-1)^2 (z+1)^2 (z-z_0)^2 \hat{Q}(z). \tag{147}$$

The function φ is a polynomial of degree 4 with leading coefficient $-f_1/4$ and

$$\varphi(1) = 4Q_1(z_0 - 1)^2 \le 0, \quad \varphi(-1) = 4Q_2(z_0 + 1)^2 \le 0, \quad \varphi(z_0) = Q_3(z_0^2 - 1)^2 \le 0. \tag{148}$$

The turning points of the geodesics are given by the roots of $\varphi(z)$. We will consider only the case where the polynomial φ has 4 distinct real roots, say α_i ($i = 1, 2, 3, 4$). Indeed, for large values of ν_1 and ν_2, we can find real roots such that either $(a) -1 < \alpha_1 < \alpha_2 < 1 < \alpha_3 < \alpha_4$, or $(b)\, \alpha_1 < \alpha_2 < -1 < \alpha_3 < \alpha_4 < 1$, depending on the values of f_a. Thus the expression for the action variable becomes the following elliptic integral

$$I_1(f) = \frac{1}{\pi} \int_{\alpha_i}^{\alpha_j} \frac{\sqrt{\varphi(z)}}{(z-1)(z+1)(z-z_0)} dz, \tag{149}$$

where α_i and α_j are the roots between -1 and +1.

It seems that real roots α_i satisfying (a) or (b) exist in the general case, although we have not proved this. One can check that real roots α_i satisfying (a) or (b) exist for small values of f_4, f_5 and any ν_i. Note that roots of the form $-1 < \alpha_1 < \alpha_2 < \alpha_3 < 1 < \alpha_4$ or $\alpha_1 < -1 < \alpha_2 < \alpha_3 < \alpha_4 < 1$ are forbidden by (148).

As a more explicit example, consider the rational limit $\nu_1, \nu_2 \to \infty$ with $\nu_2/\nu_1 = q > 1$ fixed. Keeping f_1, f_2, f_4, f_5 fixed in the limit one finds $Q_i = 0, (1 = 1, 2, 3)$ and

$$R_1 = (1 + q^2)\frac{f_2^2}{2} - \frac{f_1}{8}, \quad R_2 = -(1 + q^2)\frac{f_2^2}{2q^2} + \frac{f_1}{8}, \quad R_3 = -R_1 - R_2, \tag{150}$$

This limit is given by (192) and (193) in appendix B with $N_i = 0$. Thus the polynomial becomes

$$\begin{aligned} \varphi(z) &= (z-1)^2 (z+1)^2 (z-z_0)^2 \hat{Q}(z) \\ &\to -\frac{f_1}{4}(z-1)(z+1)(z-z_0)(z-z_1), \end{aligned} \tag{151}$$

where $z_0 = (q^2 + 1)/(q^2 - 1)$ and $z_1 = z_0(1 - 8f_2^2/f_1)$. Positivity of φ and $|z| < 1$ requires one of four cases

$$(I) \quad -1 < z < z_1 < 1 < z_0, \quad (II) \quad -1 < z < 1 < z_1 < z_0,$$
$$(III) \quad z_0 < z_1 < -1 < z < 1, \quad (IV) \quad z_0 < -1 < z_1 < z < 1. \tag{152}$$

Near the rational limit, the 4 real roots $\{-1, 1, z_0, z_1\}$ change as

$$-1 \to -1 + \delta_1, \quad 1 \to 1 + \delta_2, \quad z_0 \to z_0^*, \quad z_1 \to z_1^*, \tag{153}$$

where z_0^*, z_1^* are the new values of z_0, z_1 and δ_i are evaluated up to the order ν_i^{-2} as

$$\delta_1 = -\frac{8Q_2(1 + z_0)}{f_1(1 + z_1)}, \quad \delta_2 = \frac{8Q_1(1 - z_0)}{f_1(1 - z_1)}. \tag{154}$$

This implies the inequality $(a) -1 < \alpha_1 < \alpha_2 < 1 < \alpha_3 < \alpha_4$ or $(b)\, \alpha_1 < \alpha_2 < -1 < \alpha_3 < \alpha_4 < 1$ by $Q_i < 0$ and $f_1 > 0$:

$$(I), (II) \to (a), \quad (III), (IV) \to (b). \tag{155}$$

Therefore the action variable is of the form (149). The integral may be evaluated in the four cases

(I) $-1 < z < z_1 < 1 < z_0$:

$$|I_1| = \frac{\sqrt{f_1}(1 - z_1)}{\pi\sqrt{2(z_0 - z_1)}}(\Pi(\pi/2, (z_1 + 1)/2, \kappa) - F(\pi/2, \kappa)), \tag{156}$$

with

$$\kappa = \sqrt{\frac{(z_0 - 1)(z_1 + 1)}{2(z_0 - z_1)}}. \tag{157}$$

(II) $-1 < z < 1 < z_1 < z_0$:

$$|I_1| = \frac{\sqrt{f_1}(z_1 - 1)}{\pi\sqrt{(z_0 - 1)(z_1 + 1)}}\Pi(\pi/2, 2/(z_1 + 1), \kappa), \tag{158}$$

with

$$\kappa = \sqrt{\frac{2(z_0 - z_1)}{(z_0 - 1)(z_1 + 1)}}. \tag{159}$$

(III) $z_0 < z_1 < -1 < z < 1$:

$$|I_1| = -\frac{\sqrt{f_1}(z_1 + 1)}{\pi\sqrt{(z_0 + 1)(z_1 - 1)}}\Pi(\pi/2, 2/(1 - z_1), \kappa), \tag{160}$$

with

$$\kappa = \sqrt{\frac{2(z_1 - z_0)}{(z_0 + 1)(z_1 - 1)}}. \tag{161}$$

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(IV) $z_0 < -1 < z_1 < z < 1$:

$$|I_1| = \frac{\sqrt{f_1}(z_1 + 1)}{\pi\sqrt{2(z_1 - z_0)}}(\Pi(\pi/2, (1 - z_1)/2, \kappa) - F(\pi/2, \kappa)),$$ (162)

with

$$\kappa = \sqrt{\frac{(z_0 + 1)(z_1 - 1)}{2(z_1 - z_0)}}.$$ (163)

In these expressions F and Π denote complete elliptic integrals of the first and third kind respectively.

5.4 Slightly inhomogeneous metrics: semiclassical spectrum

In this subsection we calculate the action variable for a small perturbation about the homogeneous metrics. This is the classical computation corresponding to the spectral calculation of section 3.4 above.

The homogeneous metrics are given by $z_0 \to \infty$. In this limit

$$\varphi(z) \to z_0^2\, \hat{\varphi}(z) + o(z_0),$$ (164)

which represents the transition from the Riemann surface $y^2 = \varphi(z)$ of genus 1 to $y^2 = \hat{\varphi}(z)$ of genus 0. We therefore have

$$\hat{Q}(z) = \frac{\hat{\varphi}(z)}{(z - 1)^2(z + 1)^2},$$ (165)

where $\hat{\varphi}$ is a polynomial of degree 2:

$$\hat{\varphi}(z) = Q_1(z + 1)^2 + Q_2(z - 1)^2 + (R_1 - R_2)(z - 1)(z + 1).$$ (166)

The coefficients Q_i and R_i are given by (48), (49) and (188), again with the replacements of (145)

$$j(j + 1) \to f_2^2, \quad n_1, n_2 \to f_4, f_5, \quad \lambda \to f_1.$$ (167)

The action variable is calculated using the integral (128) to give

$$|I_1(f)| = \sqrt{|Q_1| + |Q_2| - R_1 + R_2} - \sqrt{|Q_1|} - \sqrt{|Q_2|}.$$ (168)

Solving this expression for $f_1 = H$ recovers the energy given in (132).

Now consider a perturbation away from homogeneity as in section 3.4

$$\frac{\nu_2}{\nu_1} = 1 + \epsilon, \qquad \text{with} \qquad \epsilon \ll 1.$$ (169)

In this case we have a polynomial of degree 3 by ignoring the terms that are $o(\epsilon^2)$,

$$\hat{\varphi}(z) = Q_1(z+1)^2 + Q_2(z-1)^2 + (R_1 - R_2)(z-1)(z+1) + z(z-1)(z+1)(R_1 + R_2). \quad (170)$$

The coefficients Q_i and R_i are given by (48), (49) and (183), (184) with $\nu_2/\nu_1 = 1 + \epsilon$. The last term in the polynomial (170) has coefficient $R_1 + R_2$ which is order ϵ, since $R_1 + R_2 = 0$ for $\epsilon = 0$. The integral for the action variable can be evaluated as a perturbation of the homogeneous limit. We find

$$
\begin{aligned}
|I_1(f)| &= \sqrt{|Q_1| + |Q_2| - R_1 + R_2} - \sqrt{|Q_1|} - \sqrt{|Q_2|} \\
&\quad - \frac{(R_1 + R_2)(Q_1 - Q_2)}{2(|I_1| + \sqrt{|Q_1|} + \sqrt{|Q_2|})^3} + o(\epsilon^2).
\end{aligned} \quad (171)
$$

Expressing the coefficients in terms of the homogeneous quantities I_4, I_5, namely $Q_1 = -I_4^2/4 + \epsilon Q_1^{(1)}$ and $Q_2 = -I_5^2/4 + \epsilon Q_2^{(1)}$, together with (189) and (190) in appendix B, we have

$$f_1 = f_1\Big|_{\text{homogeneous}} + \epsilon \frac{4(2 + \nu^2)}{1 + \nu^2} f_1^{(1)} + o(\epsilon^2), \quad (172)$$

where

$$
\begin{aligned}
f_1^{(1)} &= Q_1^{(1)} + Q_2^{(1)} + R_1^{(1)} - R_2^{(1)} - (2|I_1| + |I_4| + |I_5|)\left(\frac{Q_1^{(1)}}{|I_4|} + \frac{Q_2^{(1)}}{|I_5|}\right) \\
&\quad - \frac{(I_4^2 - I_5^2)(R_1^{(1)} + R_2^{(1)})}{(2|I_1| + |I_4| + |I_5|)^2}.
\end{aligned} \quad (173)
$$

The perturbed energy is just $H = f_1$. Therefore the semiclassical spectrum is computed by setting $|I_1| \to L$, $I_4 \to N_1$ and $I_5 \to N_2$ in equations (172) and (173). It is satisfying to see that the resulting semiclassical spectrum agrees with the Laplacian spectrum (65) within a semi-classical approximation. This provides a check on our calculations. We can understand why the first term in (65) is lacking from the semiclassical result by again comparing a naive quantisation of the classical Hamiltonian (137), $p_a \to i\partial/\partial x^a$, with the Laplacian (25). There is an ordering ambiguity, and we see that the $P(z)d/dz$ term in the Fuchs equation does not exist in the quantiation of the classical Hamiltonian.

6 Conclusions and future directions

We have studied the Laplacian spectrum, Lichnerowicz spectrum and geodesics on an infinite family of five dimensional inhomogeneous Einstein metrics. The moduli space of metrics included an infinite sequence of homogeneous metrics, $T^{p,1}$. For the homogeneous metrics, we were able to give very explicit results for the Laplacian spectrum and for the frequences

of closed geodesics. Further, we were able to use the explicit spectrum on $T^{p,1}$ to calculate the zeta function on these metrics at large p.

The inhomogeneous metrics are harder to study. We have shown that the Laplace equation and Hamilton-Jacobi equation may be separated for these cases. We found some perturbative results for the Laplacian spectrum for slightly inhomogeneous spaces. However, it seems that the full inhomogeneous spectrum will require numerical calculations or more sophisticated methods than we have used.

For the geodesics in the inhomogeneous case we have identified four of the five action variables. The calculation of the remaining action variable reduces to an elliptic integral with an underlying elliptic curve of genus one. If one could calculate the remaining action variable, then it is possible that the semiclassical quantisation of the system would give some insight into the Laplacian spectrum for the inhomogeneous metrics. Semiclassical quantisation of the homogeneous and slightly inhomogeneous cases gives a good agreement with the Laplacian spectrum.

The Lichnerowicz spectrum contains information about the stability of Freund-Rubin compactifications and generalised black holes constructed from Einstein metrics. We saw that of the family of metrics we considered, only $T^{1,1}$ can be shown to give a stable Freund-Rubin compactification and generalised black holes with vanishing cosmological constant. We suspect that the remaining spacetimes are all unstable. Unstable generalised black holes are interesting as it is unclear what the endpoint of the instability will be and the instability may give rise to interesting dynamics, analogous to what has been discovered recently for the black string instability [18].

We saw that the moduli space contains metrics with an arbitrarily large maximum of Weyl eigenvalues. This occurs in the rational limit at large inhomogeneity q, see Table 1 above. This fact suggests that the minimum Lichnerowicz eigenvalue may become arbitrarily negative [8], giving rise to unstable generalised Anti-de Sitter black holes. The instability of these black holes predicts a thermal instability of a dual theory propagating on the corresponding Einstein metric [13, 12]. If one could calculate the zeta function for these backgrounds, the corresponding thermodynamics may help elucidate the nature of the predicted field theory instability. Perhaps the Laplacian spectrum in the large-q rational limit is calculable? One should note that it is not certain that the dual instability will necessarily be present for a free scalar field, as duality relates the AdS_7 black hole to the the strongly coupled thermal theory living on $M5$ branes which is certainly much more complicated. However, it seems possible that a thermodynamic instability for field theories

on a curved background with regions of large curvature is a generic phenomenon.

Finally, it would be interesting to perform an analysis similar to ours for other Einstein metrics. In particular, Einstein-Sasaki metrics always give stable Freund-Rubin compactifications and stable generalised black holes [8]. Therefore a study of the Laplacian spectrum of the five dimensional Einstein-Sasaki metrics constructed in [9] may have interesting applications to the AdS/CFT correspondence. These metrics are also cohomogeneity one and it seems clear that the equations can also be separated in these cases.

Acknowledgements

A   Metric terms

We normalise the metrics so that the Ricci scalar is $R = 20$.

$$h^2(\theta) = \frac{(1 - \nu_1^2 \nu_2^2)}{(2 - \nu_1^2 - \nu_2^2)} \frac{1 - \nu_1^2 \cos^2 \theta - \nu_2^2 \sin^2 \theta}{1 - \mu_1^2 \cos^2 \theta - \mu_2^2 \sin^2 \theta}, \tag{174}$$

$$a_{11}(\theta) = \frac{\nu_1^2 (1 - \nu_2^2)^2 (2 - \nu_1^2 - \nu_2^2)(1 - \nu_1^2 \nu_2^2)}{4(1 + \nu_1^4 \nu_2^2 + \nu_1^2 \nu_2^4 - 3\nu_1^2 \nu_2^2)^2} \frac{(1 - \mu_1^2 \cos^2 \theta - \nu_2^2 \sin^2 \theta) \sin^2 \theta}{1 - \nu_1^2 \cos^2 \theta - \nu_2^2 \sin^2 \theta}, \tag{175}$$

$$a_{22}(\theta) = \frac{\nu_2^2 (1 - \nu_1^2)^2 (2 - \nu_1^2 - \nu_2^2)(1 - \nu_1^2 \nu_2^2)}{4(1 + \nu_1^4 \nu_2^2 + \nu_1^2 \nu_2^4 - 3\nu_1^2 \nu_2^2)^2} \frac{(1 - \nu_1^2 \cos^2 \theta - \mu_2^2 \sin^2 \theta) \cos^2 \theta}{1 - \nu_1^2 \cos^2 \theta - \nu_2^2 \sin^2 \theta}, \tag{176}$$

$$a_{12}(\theta) = -\frac{\nu_1^2 \nu_2^2 (1 - \nu_1^2)^2 (1 - \nu_2^2)^2 (1 - \nu_1^2 \nu_2^2)}{4(1 + \nu_1^4 \nu_2^2 + \nu_1^2 \nu_2^4 - 3\nu_1^2 \nu_2^2)^2} \frac{\sin^2 \theta \cos^2 \theta}{1 - \nu_1^2 \cos^2 \theta - \nu_2^2 \sin^2 \theta}, \tag{177}$$

$$b^2(\theta) = \frac{(1 - \nu_1^2 \nu_2^2)(1 - \nu_1^2 \cos^2 \theta - \nu_2^2 \sin^2 \theta)}{4(1 + \nu_1^4 \nu_2^2 + \nu_1^2 \nu_2^4 - 3\nu_1^2 \nu_2^2)}, \tag{178}$$

with

$$\mu_1^2 = \frac{\nu_1^2 (1 - \nu_1^2 \nu_2^2)}{2 - \nu_1^2 - \nu_2^2}, \qquad \mu_2^2 = \frac{\nu_2^2 (1 - \nu_1^2 \nu_2^2)}{2 - \nu_1^2 - \nu_2^2}. \tag{179}$$

The metric has the following integrality conditions

$$k_1 = \frac{\nu_1 (1 - \nu_2^2)(2 - \nu_2^2 - \nu_1^2 \nu_2^2)}{1 + \nu_1^4 \nu_2^2 + \nu_1^2 \nu_2^4 - 3\nu_1^2 \nu_2^2} \in \mathbb{Z}, \tag{180}$$

$$k_2 = \frac{\nu_2 (1 - \nu_1^2)(2 - \nu_1^2 - \nu_1^2 \nu_2^2)}{1 + \nu_1^4 \nu_2^2 + \nu_1^2 \nu_2^4 - 3\nu_1^2 \nu_2^2} \in \mathbb{Z}. \tag{181}$$

In the homogeneous case, $\nu_1 = \nu_2 = \nu$,

$$
\begin{aligned}
a^{11} &= \frac{2(2\nu^2+1)^2}{(\nu^2+2)(1+\nu^2)}\left(1+\frac{2}{\nu^2\sin^2\theta}\right), \\
a^{22} &= \frac{2(2\nu^2+1)^2}{(\nu^2+2)(1+\nu^2)}\left(1+\frac{2}{\nu^2\cos^2\theta}\right), \\
a^{12} &= \frac{2(2\nu^2+1)^2}{(\nu^2+2)(1+\nu^2)}, \\
b^{-2} &= \frac{4(2\nu^2+1)}{1+\nu^2}, \quad h^{-2} = \frac{\nu^2+2}{1+\nu^2}, \quad \sqrt{\det a} \propto \sin\theta\cos\theta.
\end{aligned}
\tag{182}
$$

B Coefficients in the inhomogeneous equation

The coefficients R_i are given by

$$
\begin{aligned}
R_1 &= \frac{(j(j+1)-(n_1+n_2)^2)B}{8(1-\mu_1^2)} - \frac{A_1 r_+ n_1^2}{2} + \frac{A_2(1-\nu_1^2)n_2^2}{8(1-\mu_1^2)} + \frac{A_{12}(1-\nu_1^2)n_1 n_2}{4(1-\mu_1^2)^2} \\
&\quad - \frac{(1-\nu_1^2)(1-\nu_1^2\nu_2^2)\lambda}{8(2-\nu_1^2-\nu_2^2)(1-\mu_1^2)},
\end{aligned}
\tag{183}
$$

$$
\begin{aligned}
R_2 &= -\frac{(j(j+1)-(n_1+n_2)^2)B}{8(1-\mu_2^2)} + \frac{A_2 r_- n_2^2}{2} - \frac{A_1(1-\nu_2^2)n_1^2}{8(1-\mu_2^2)} - \frac{A_{12}(1-\nu_2^2)n_1 n_2}{4(1-\mu_2^2)^2} \\
&\quad + \frac{(1-\nu_2^2)(1-\nu_1^2\nu_2^2)\lambda}{8(2-\nu_1^2-\nu_2^2)(1-\mu_2^2)}, \\
R_3 &= -R_1 - R_2,
\end{aligned}
\tag{184}
$$

where

$$
\begin{aligned}
A_1 &= \frac{4(1+\nu_1^4\nu_2^2+\nu_1^2\nu_2^4-3\nu_1^2\nu_2^2)^2}{\nu_1^2(1-\nu_2^2)^2(2-\nu_1^2-\nu_2^2)^2}, \\
A_2 &= \frac{4(1+\nu_1^4\nu_2^2+\nu_1^2\nu_2^4-3\nu_1^2\nu_2^2)^2}{\nu_2^2(1-\nu_1^2)^2(2-\nu_1^2-\nu_2^2)^2}, \\
A_{12} &= \frac{4(1+\nu_1^4\nu_2^2+\nu_1^2\nu_2^4-3\nu_1^2\nu_2^2)^2}{(2-\nu_1^2-\nu_2^2)^3}, \\
B &= \frac{4(1+\nu_1^4\nu_2^2+\nu_1^2\nu_2^4-3\nu_1^2\nu_2^2)}{2-\nu_1^2-\nu_2^2},
\end{aligned}
\tag{185}
$$

and

$$
\begin{aligned}
r_+ &= -\frac{c_1(1-\nu_1^2)^2}{(1-\mu_1^2)^3} + \frac{c_2(1-\nu_1^2)}{2(1-\mu_1^2)^2} + \frac{c_3(1-\nu_1^2)}{2(1-\mu_1^2)^2} - \frac{(1-\nu_1^2)^2}{4(1-\mu_1^2)^2}, \\
r_- &= \frac{c_1(1-\nu_2^2)^2}{(1-\mu_2^2)^3} - \frac{c_2(1-\nu_2^2)}{2(1-\mu_2^2)^2} - \frac{c_4(1-\nu_2^2)}{2(1-\mu_2^2)^2} - \frac{(1-\nu_2^2)^2}{4(1-\mu_2^2)^2},
\end{aligned}
\tag{186}
$$

with

$$c_1 = -\frac{(\nu_1^2 - \nu_2^2)(1 - \nu_1^2\nu_2^2)}{2(1 - \nu_1^2 - \nu_2^2)},$$

$$c_2 = \frac{-\nu_1^2 + \nu_2^2}{2},$$

$$c_3 = \frac{-2\nu_1^2 + \nu_2^2 + \nu_1^4 + \nu_1^2\nu_2^2 - \nu_1^2\nu_2^4}{2(2 - \nu_1^2 - \nu_2^2)},$$

$$c_4 = \frac{2\nu_2^2 - \nu_1^2 - \nu_2^4 - \nu_1^2\nu_2^2 + \nu_1^4\nu_2^2}{2(2 - \nu_1^2 - \nu_2^2)}. \tag{187}$$

In the homogeneous case, $\nu_1 = \nu_2 = \nu$, we have

$$
\begin{aligned}
R_1 &= -R_2 \\
&= \frac{(1 + 2\nu^2)(j(j+1) - (n_1 + n_2)^2)}{2(2 + \nu^2)} + \frac{(1 + 2\nu^2)^2(n_1^2 + n_2^2)}{4\nu^2(2 + \nu^2)} + \frac{(1 + 2\nu^2)^2 n_1 n_2}{2(2 + \nu^2)^2} - \frac{(1 + \nu^2)\lambda}{8(2 + \nu^2)}, \\
R_3 &= 0.
\end{aligned} \tag{188}
$$

We also collect here the terms for the perturbation about the homogeneous metrics. The required terms for $Q^{(1)}$ are

$$
\begin{aligned}
Q_1^{(1)} &= \frac{4(2\nu^2 + 1)(\nu^4 + 2\nu^2 + 3)}{(\nu^2 + 2)^3(\nu^2 - 1)} n_1^2, \\
Q_2^{(1)} &= -\frac{2(2\nu^2 + 1)(7\nu^4 + 3\nu^2 + 2)}{(\nu^2 + 2)^3(\nu^2 - 1)\nu^2} n_2^2,
\end{aligned} \tag{189}
$$

whilst the terms for $R^{(1)}$ are

$$
\begin{aligned}
R_1^{(1)} &= \frac{\nu^2(-1 + 3\nu^2 + \nu^4)}{(\nu^2 - 1)(\nu^2 + 2)^2}(j(j+1) - (n_1 + n_2)^2) - \frac{20 + 56\nu^2 + 37\nu^4 + 8\nu^6 - 4\nu^8}{4(\nu^2 - 1)(\nu^2 + 2)^3} n_1^2 \\
&\quad + \frac{4 + 10\nu^2 + 11\nu^4 + 16\nu^6 + 4\nu^8}{4\nu^2(\nu^2 - 1)(\nu^2 + 2)^2} n_2^2 + \frac{\nu^2(-6 - 5\nu^2 + 16\nu^4 + 4\nu^6)}{2(\nu^2 - 1)(\nu^2 + 2)^3} n_1 n_2 \\
&\quad + \frac{\nu^2}{4(\nu^2 - 1)(\nu^2 + 2)^2}\lambda^{(0)}, \\
R_2^{(1)} &= \frac{\nu^2(2 + \nu^4)}{(\nu^2 - 1)(\nu^2 + 2)^2}(j(j+1) - (n_1 + n_2)^2) + \frac{10 + 23\nu^2 + 8\nu^4 + 4\nu^6}{4(\nu^2 - 1)(\nu^2 + 2)^2} n_1^2 \\
&\quad - \frac{8 + 24\nu^2 + 32\nu^4 + 41\nu^6 + 16\nu^8 - 4\nu^{10}}{4\nu^2(\nu^2 - 1)(\nu^2 + 2)^3} n_2^2 + \frac{\nu^2(6 + 7\nu^2 - 8\nu^4 + 4\nu^6)}{2(\nu^2 - 1)(\nu^2 + 2)^3} n_1 n_2 \\
&\quad + \frac{\nu^4}{4(\nu^2 - 1)(\nu^2 + 2)^2}\lambda^{(0)}.
\end{aligned} \tag{190}
$$

Finally, the coefficients simplify in the rational limit, $\nu_1, \nu_2 \to \infty$ with $\nu_2/\nu_1 = q > 1$ fixed. One obtains

$$z_0 = \frac{q^2 + 1}{q^2 - 1}. \tag{191}$$

The coefficients (46) become, introducing the integers N_i by $n_i = k_i N_i/2$,

$$Q_1 = -\frac{(1+q^2)^2}{16q^4}N_1^2, \quad Q_2 = -\frac{(1+q^2)^2}{16q^2}N_2^2, \quad Q_3 = -\frac{1+q^2}{16q^2}(N_1+N_2)^2. \tag{192}$$

Furthermore,

$$
\begin{aligned}
R_1 &= (1+q^2)\frac{j(j+1)}{2} - \frac{\lambda}{8} \\
&+ \frac{1}{32q^4}(1+3q^2-q^4-2q^6)N_1^2 + \frac{q^2}{32}N_2^2 - \frac{2+q^2}{16}N_1N_2, \\
R_2 &= -(1+q^2)\frac{j(j+1)}{2q^2} + \frac{\lambda}{8} \\
&- \frac{1}{32q^4}N_1^2 + \frac{1}{32q^4}(2+q^2-3q^4-q^6)N_2^2 + \frac{1}{16q^4}(1+2q^2)N_1N_2, \\
R_3 &= -R_1 - R_2. \tag{193}
\end{aligned}
$$

C Properties of Jacobi polynomials

All the following identities involving Jacobi polynomials may be found, for example, in [19] section 8.96.

The first equation we use allows us to express $zP_n^{(\alpha,\beta)}(z)$ in terms of $P_n^{(\alpha,\beta)}(z)$, $P_{n-1}^{(\alpha,\beta)}(z)$ and $P_{n+1}^{(\alpha,\beta)}(z)$

$$
\begin{aligned}
& 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(z) \\
&= (2n+\alpha+\beta+1)[(2n+\alpha+\beta)(2n+\alpha+\beta+2)z+\alpha^2-\beta^2]P_n^{(\alpha,\beta)}(z) \\
&- 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(z). \tag{194}
\end{aligned}
$$

The following equation allows us to express $(1-z^2)dP_n^{(\alpha,\beta)}(z)/dz$ in terms of $P_n^{(\alpha,\beta)}(z)$, $zP_n^{(\alpha,\beta)}(z)$ and $P_{n-1}^{(\alpha,\beta)}(z)$

$$
\begin{aligned}
& (2n+\alpha+\beta)(1-z^2)\frac{d}{dz}P_n^{(\alpha,\beta)}(z) \\
&= n[(\alpha-\beta)-(2n+\alpha+\beta)z]P_n^{(\alpha,\beta)}(z) + 2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(z). \tag{195}
\end{aligned}
$$

Once we have used the above relations, we use two integration results. The first is

$$
\begin{aligned}
& \int_{-1}^{1}(1-z)^\alpha(1+z)^\beta P_m^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(z)dz \\
&= 0, \qquad\qquad\qquad\qquad\qquad\quad \text{if} \quad m \neq n \\
&= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+1+2n)\Gamma(\alpha+\beta+n+1)}, \quad \text{if} \quad m = n. \tag{196}
\end{aligned}
$$

The second relation we will use is

$$\int_{-1}^{1}(1-z)^{\alpha-1}(1+z)^\beta P_n^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(z)dz = \frac{2^{\alpha+\beta}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,\alpha\,\Gamma(\alpha+\beta+n+1)}. \tag{197}$$

D   Check of the zeta function

The zeta function on an n dimensional compact manifold (in our case $n = 5$),

$$\zeta(s) = \sum_{i;\lambda_i > 0} \lambda_i^{-s}, \tag{198}$$

can be shown to converge absolutely in the region $\mathfrak{Re}\,s > n/2$, and can be analytically extended to a meromorphic function of s in the whole complex plane. The poles are located at $s = n/2 - k$, $(k = 0, 1, 2, \cdots)$ and their residues are given by

$$\mathrm{Res}_{s=n/2-k} = \frac{a_{2k}}{(4\pi)^{n/2}\Gamma(n/2 - k)}. \tag{199}$$

If the operator is $A = -\Box + \kappa$ then the first few a_k are given by

$$a_0 = \mathrm{vol}(M), \quad a_2 = -a_0\,\kappa + \frac{1}{6}\int_M dx R, \tag{200}$$

and

$$a_4 = \int_M dx\left(\frac{1}{180}R^{abcd}R_{abcd} - \frac{1}{180}R^{ab}R_{ab} + \frac{1}{72}R^2 - \frac{1}{30}\nabla_a\nabla^a R\right). \tag{201}$$

Is our computation of the homogeneous zeta function able to reproduce any of these results? We have just recalled that there should be a pole at $s = 5/2$ with residue

$$\mathrm{Res}_{s=5/2} = \frac{\mathrm{vol}(M)}{(4\pi)^{5/2}\Gamma(5/2)}. \tag{202}$$

In the large ν limit, this gives

$$\mathrm{Res}_{s=5/2} = \frac{\sqrt{2}}{24\nu}. \tag{203}$$

We can compare this with our results. We have already noted that case I is always subleading compared to cases III and IV. We see that cases III and IV do have a pole at $s = 5/2$. The total residue of $\zeta(s)$ in equation (73) turns out to be

$$\mathrm{Res}_{s=5/2} = \frac{\sqrt{2}}{24\nu}. \tag{204}$$

Which is precisely as required! We see that case II cannot contribute to this expression because it has no terms that are $\mathcal{O}(1/\nu)$. Thus we have a rather nontrivial check on our calculation.

It is not possible to check the other residues because we have only worked to leading order in ν. The other poles do not appear to this order.

E Heun's equation

In order to transform the Fuchsian equation of section 3.3, equations (44) to (46), to a canonical form write

$$\phi(z) = (1 - z)^{\sqrt{|Q_1|}}(1 + z)^{\sqrt{|Q_2|}}(z - z_0)^{\sqrt{|Q_3|}} f(z). \tag{205}$$

The new variable $f(x)$ with $x = (1 - z)/2$ satisfies Heun's equation

$$\frac{d^2}{dx^2}f + \left(\frac{\gamma}{x} + \frac{\delta}{x - 1} + \frac{\epsilon}{x - x_0}\right)\frac{d}{dx}f + \frac{\alpha\beta x - q}{x(x - 1)(x - x_0)}f = 0, \tag{206}$$

where

$$\alpha + \beta = 2\left(1 + \sum_{i=1}^{3}\sqrt{|Q_i|}\right),$$

$$\alpha\beta = 2\left(\sum_{i=1}^{3}\sqrt{|Q_i|} + \sum_{i<j}\sqrt{|Q_iQ_j|} + (R_1 + R_2)x_0 - R_2\right),$$

$$\gamma = 1 + 2\sqrt{|Q_1|},$$

$$\delta = 1 + 2\sqrt{|Q_2|},$$

$$\epsilon = 1 + 2\sqrt{|Q_3|},$$

$$q = (\sqrt{|Q_1|} + \sqrt{|Q_2|} + 2\sqrt{|Q_1Q_2|})x_0 + \sqrt{|Q_1|} + \sqrt{|Q_3|} + 2\sqrt{|Q_1Q_3|} + 2R_1x_0 \tag{207}$$

One has

$$1 + \alpha + \beta - \gamma - \delta - \epsilon = 0. \tag{208}$$

It is known that Huen's equation admits an expression in terms of elliptic functions and this expression is closely related to the Inozemtsev system. For example see [21, 20]. From these references, if all of γ, δ, ϵ are half-odd-integer one can obtain exact solutions of Huen's equation. Unfortunately in our case this condition is not satisfied: γ, δ are integers and $\epsilon \in \mathbb{R}$.

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